Congruences on the product of two full transformation monoids

Wolfram Bentz

University of Hull

Joint work with João Araújo (Universade Aberta/CEMAT) and Gracinda M.S. Gomes (University of Lisbon/CEMAT)

NBSAN York

January 18, 2017

- For $n \in \mathbb{N}$, let $X_n = \{1, \ldots, n\}$.
- Let $\mathcal{T}_n = X_n^{X_n}$ be full transformation monoid, $\mathcal{PT}_n$ the full partial transformation monoid, and $\mathcal{I}_n$ the symmetric monoid, all on the set X_n.
- The congruences of these monoids have been described over 50 years ago by Malcev ($\mathcal{T}_n$), Sutov ($\mathcal{PT}_n$), and Liber ($\mathcal{I}_n$).
- Now consider the product monoids $\mathcal{T}_m \times \mathcal{T}_n$.
- Surprisingly, we could not find a description of $\text{Con}(\mathcal{T}_m \times \mathcal{T}_n)$.
- Our work describes those congruences.

Let Q_n be one of $\mathcal{T}_n, \mathcal{PT}_n, \mathcal{I}_n$.
Given $f \in Q_n$ we denote its domain by $\operatorname{dom}(f)$, its image by $\operatorname{im}(f)$, its kernel by $\ker(f)$ and its rank (the size of the image of f) by $|f|$.

Lemma

Let $f, g \in Q_n$. Then

1. $f\mathcal{D}g$ *iff* $|f| = |g|$;
2. $f\mathcal{L}g$ *iff f and g have the same image;*
3. $f\mathcal{R}g$ *iff f and g have the same domain and kernel;*
4. $f\mathcal{H}g$ *iff f and g have the same domain, kernel, and image.*

Theorem (after Ganyushkin and Mazorchuk)

A non-universal congruence of Q_n is associated with a pair (k, N), where $1 \leq k \leq n$, and N is a normal subgroup of S_k; and it is of the form $\theta(k, N)$ defined as follows: for all $f, g \in Q_n$,

$$f \, \theta(k,N) \, g \quad \text{iff} \quad \begin{cases} f = g \text{ and } |f| > k, \text{ or} \\ |f|, |g| < k, \text{ or} \\ |f| = |g| = k, f \mathcal{H} g \text{ and } f = g \cdot \omega, \text{ where } \omega \in N. \end{cases}$$

We can rephrase the congruence theorem as follows.
Let θ be a congruence on Q_n. Then the $\mathcal{D}$-classes of Q_n partition into the following three types (up to 2 of which might not be present)

1. A set $\{D_i, D_{i+1}, \dots, D_n\}$ "at the top", where $[x]_\theta = \{x\}$.
2. A set $\{D_j, D_{j-1}, \dots, D_{1/0}\}$ "at the bottom", where $[x]_\theta = D_j \cup \dots \cup D_{1/0}$.
3. A singleton $\{D_k\}$ in between those, where "something different" happens.

In the last case, the blocks of $\theta|_{D_k}$ will consists of either $\mathcal{H}$-classes, "half"-$\mathcal{H}$-classes, corresponding to odd and even permutation, or for $k = 4$, of "sixth"-$\mathcal{H}$-classes, corresponding to cosets of V in S_4.

- To make this more precise, let $\sigma \in N$, a normal subgroup of S_i.
- For each $\mathcal{H}$-class fix orderings of its kernel classes and image elements. Let f be in $\mathcal{H}$-class H.
- Then $f \cdot \sigma$ is the element of H that maps kernel K_j to image $i_{j\sigma}$, where the indices come from the orderings.
- This action will only be used to define congruence classes.
- In this context, normality will guarantee that the initial ordering does not matter.

Definition

A *θ-dlock* X is a non-empty subset of Q_n such that

1. X is a union of θ-classes as well as a union of $\mathcal{D}$-classes;
2. No proper non-empty subset of X satisfies (**??**).

Definition

Let X be a θ-dlock. We say that X has *type*

- ε if for all $a, b \in X$ with $a \, \theta \, b$ we have $a = b$;
- $\mathcal{H}$ if there exist $a, b \in X$ such that $a \neq b$ and $a \, \theta \, b$, and for all $a', b' \in X$ such that $a' \, \theta \, b'$ we have $a' \mathcal{H} b'$;
- F if there exist $a, b \in X$ such that $(a, b) \notin \mathcal{H}$ and $a \, \theta \, b$.

The congruence theorem now says that every $\theta \in \mathrm{Con}(Q_n)$ has an optional F-dlock $D_{0/1} \cup \cdots \cup D_k$, an optional $\mathcal{H}$-block D_{k+1}, sitting on top of the F-dlock, and consists otherwise of ε-dlocks.

Symmetric groups

Theorem (Folklore)

The normal subgroups N of $S_i \times S_k$ are exactly the ones of the following form:

1. $N = N_1 \times N_2$, *where* $N_1 \trianglelefteq S_i, N_2 \trianglelefteq S_k$.
2. $N = \{(a, b) \in S_i \times S_j \mid a \text{ and } b \text{ have the same parity}\}$.

The product $\mathcal{T}_m \times \mathcal{T}_n$

Similar to the group case, the congruence structure of $\mathcal{T}_m$ and T_n carries over to some degree to their product. The problem is the combinatorial complexity of this correspondence.

Recall that the $\mathcal{D}$-classes of $\mathcal{T}_m \times \mathcal{T}_n$ are exactly the products of the $\mathcal{D}$-classes of each factor. We will let $D_{i,j}$ denote that the $\mathcal{D}$-class of all (f, g) with $|f| = i$ and $|g| = j$, and for a suitable index set P, we set $D_P = \cup_{(i,j)\in P}$.

Definition

A *θ-dlock* X is a non-empty subset of $\mathcal{T}_m \times \mathcal{T}_n$ such that

1. X is a union of θ-classes as well as a union of $\mathcal{D}$-classes;
2. No proper non-empty subset of X satisfies (**??**).

Definition

Let X be a θ-dlock. We say that X has *first component type*

- ε if for all $(a,b),(c,d) \in X$ with $(a,b)\,\theta\,(c,d)$ we have $a = c$;
- $\mathcal{H}$ if there exist $(a,b),(c,d) \in X$ such that $a \neq c$ and $(a,c)\,\theta\,(b,d)$, and for all (a',b'), $(c',d') \in X$ such that $(a',b')\,\theta\,(c',d')$ we have $a'\mathcal{H}c'$;
- F if there exist $(a,b),(c,d) \in X$ such that $(a,c) \notin \mathcal{H}$ and $(a,c)\,\theta\ (b,d)$.

We define the *second component type* of X dually. Finally we say that X has *type* VW for $V, W \in \{\varepsilon, \mathcal{H}, F\}$ if it has first component type V and second component type W.

Now let θ be a congruence of $\mathcal{T}_m \times \mathcal{T}_n$, where $m, n > 1$. Then θ has up to 9 different type of dlocks. We now describe which dlocks can occur in which configurations, and how θ is defined on each dlock, giving a complete description of all congruences.

Lemma

Let $X = D_P = \cup_{(i,j)\in P} D_{i,j}$ be a θ-dlock of type FF. Then P is a downward-closed subset of $(\{1,\ldots,m\} \times \{1,\ldots,n\}, \leq \times \leq)$, and X is a single θ-class that is an ideal of $\mathcal{T}_m \times \mathcal{T}_n$.
Conversely, let θ have an ideal class I, and let

$$P \subseteq \{1,\ldots,m\} \times \{1,\ldots,n\}$$

be the set of pairs (i,j) for which $D_{i,j} \cap I \neq \emptyset$. Then I is a dlock of Type FF.

It follows that θ has at most one dlock of type FF.

Dlock of type *FF*

We can visualize a dlock of type *FF* as a “landscape”.

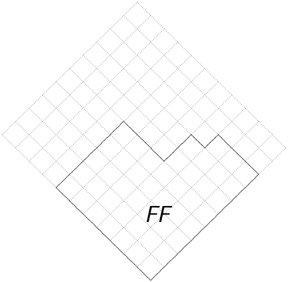


Figure: A possible configuration for a dlock of type *FF*

Lemma

Let X be a θ-dlock. Then X is of type εF or $\mathcal{H}F$ if and only if there exist $1 \leq i \leq m$, $1 \leq j \leq n$, and $N \trianglelefteq S_i$, such that $X = D_P$ with $P = \{i\} \times \{1, \ldots, j\}$, and for every $(f, g) \in X$,

$$[(f,g)]_\theta = \{(f', g') \in X | f\mathcal{H}f' \text{ and } f' = f \cdot \sigma \text{ for some } \sigma \in N\}. \qquad (1)$$

If X satisfies these requirements, then X is of type $\mathcal{H}F$ exactly when $N \neq \varepsilon_i$.

If X is a dlock of type $\mathcal{H}F$ or εF, we call the group $N \trianglelefteq S_i$ from Lemma **??** the *normal subgroup associated with X*. Clearly, a dual version of Lemma **??** holds for dlocks X of type $F\varepsilon$ or $F\mathcal{H}$.

Dlocks of type $\mathcal{H}F$

Lemma

Let $X = D_P$ be a dlock of type $\mathcal{H}F$ with $P = \{i\} \times \{1, \ldots, j\}$ and associated normal subgroup $N \trianglelefteq S_i$. Then $i \geq 2$, and $\{1, \ldots, i-1\} \times \{1, \ldots, j\}$ is contained in the index set of a θ-dlock of type FF.

Moreover, every congruence θ has at most one dlock of type $\mathcal{H}F$.

It follows that θ has an $\mathcal{H}F$-dlock, then it also has an FF-dlock.

Dlocks of type $\mathcal{H}F$ and $F\mathcal{H}$

We may visualize the statement of Lemma **??** by saying that a θ-dlock of type $\mathcal{H}F$ must lies on the "most eastern slope" of the dlock of type FF.

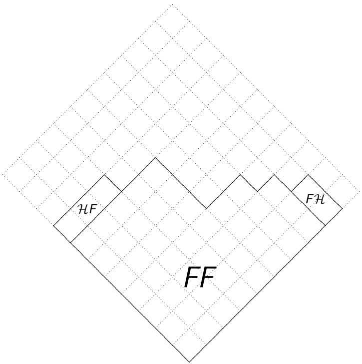


Figure: A possible configuration for three dlocks of type FF, $\mathcal{H}F$ and $F\mathcal{H}$

Dlocks of type εF

Lemma

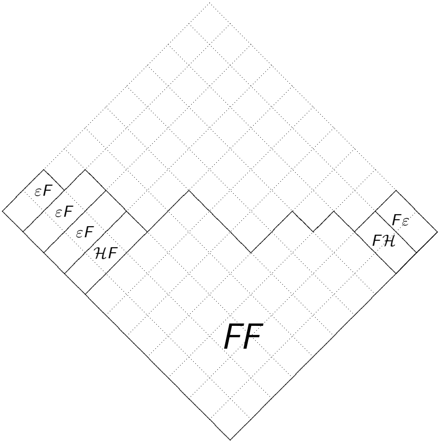
Let $X = D_P$ be a θ-dlock of type εF with $P = \{i\} \times \{1, \ldots, j\}$. Then for each $k = 1, \ldots, i-1$, the set $\{k\} \times \{1, \ldots, j\}$ is contained in the index set of a θ-dlock of type FF, $\mathcal{H}F$, or εF.

Lemma

For a given congruence θ, let J be the set of values (i, j_i) such that $\{i\} \times \{1, \ldots, j_i\}$ is the index set of a θ-dlock of type εF. Then $\pi_1(J)$ is a set of consecutive integers (possibly empty), and the values j_i are non-increasing in i.

Dlocks of type εF and $F\varepsilon$

θ-dlocks of type εF, are layered on the top of each other without "overhanging", with the lowest one either starting at $i = 1$, lying on the top of a dlock of type FF, or lying on the top of a dlock of type $\mathcal{H}F$.



Lemma

Let X be a dlock. Then X is of type $\mathcal{H}\mathcal{H}$, $\varepsilon\mathcal{H}$, $\mathcal{H}\varepsilon$, or $\varepsilon\varepsilon$ if and only if $X = D_{i,j}$ for some i, j and there exists $N \trianglelefteq S_i \times S_j$ such that for every $(f, g) \in X$,

$$[(f,g)]_\theta = \{(f',g') \in D_{i,j} \mid f\mathcal{H}f', g\mathcal{H}g', \text{ and } (f',g') = (f\cdot\sigma, g\cdot\tau) \text{ for some } (\sigma,\tau) \in N\}.$$

Moreover, in this situation,

(a) *X is of type $\mathcal{H}\mathcal{H} \iff \pi_1(N) \neq \varepsilon_i$ and $\pi_2(N) \neq \varepsilon_j$;*

(b) *X is of type $\varepsilon\mathcal{H} \iff N = \varepsilon_i \times N'$ for some $N' \neq \varepsilon_j$;*

(c) *X is of type $\mathcal{H}\varepsilon \iff N = N' \times \varepsilon_j$ for some $N' \neq \varepsilon_i$;*

(d) *X is of type $\varepsilon\varepsilon \iff N = \varepsilon_i \times \varepsilon_j$.*

Dlocks of type $\mathcal{H}\mathcal{H}$

Lemma

Let $X = D_{i,j}$ be a θ-dlock of type $\mathcal{H}\mathcal{H}$ with normal subgroup $N \trianglelefteq S_i \times S_j$. Then $i, j \geq 2$, and $D_{i,j-1}$ is contained in either a θ-dlock of type FF, or in a θ-dlock of type $\mathcal{H}F$ with normal subgroup $N' \trianglelefteq S_i$, where $\pi_1(N) \subseteq N'$. Symmetrically, $D_{i-1,j}$ is contained in either a θ-dlock of type FF, or in a θ-dlock of type $F\mathcal{H}$ with normal subgroup $N' \trianglelefteq S_j$, where $\pi_2(N) \subseteq N'$.

Dlocks of type $\mathcal{HH}$

The result means that the dlocks of type $\mathcal{HH}$ can only occupy the "valleys" in the landscape formed by the dlocks of FF, $\mathcal{H}F$, and $F\mathcal{H}$.

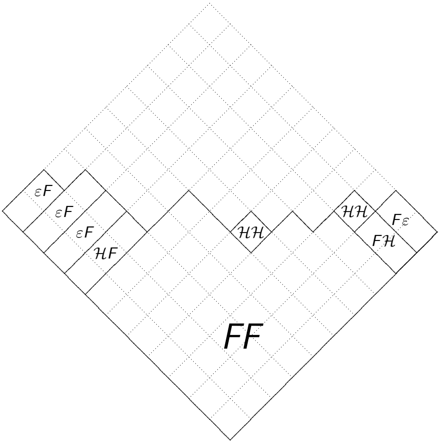


Figure: A possible configuration for dlocks of type FF, $\mathcal{H}F$, $F\mathcal{H}$, εF, $F\varepsilon$, and $\mathcal{HH}$

Dlocks of type εF

Lemma

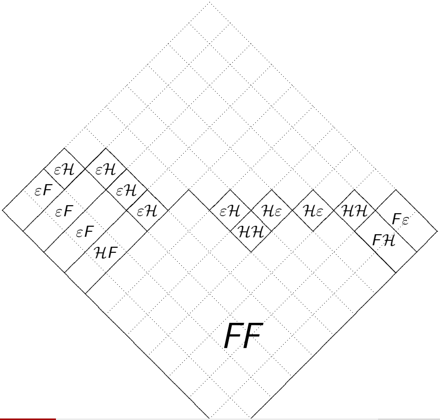
Let $X = D_{i,j}$ be a θ-dlock of type $\varepsilon\mathcal{H}$ with normal subgroup $N = \varepsilon_i \times N' \trianglelefteq S_i \times S_j$. Then $j \geq 2$, and $D_{i,j-1}$ is contained in a dlock of type FF, $\mathcal{H}F$, or εF.

Moreover, if $i > 1$, then $D_{i-1,j}$ is contained in one of the following:

1. *a dlock of type FF;*
2. *a dlock of type $\mathcal{H}F$;*
3. *a dlock of type εF;*
4. *a dlock of type $F\mathcal{H}$ with normal subgroup $\bar{N} \trianglelefteq S_j$ such that $N' \subseteq \bar{N}$;*
5. *a dlock of type $\mathcal{H}\mathcal{H}$ with normal subgroup $\bar{N} \trianglelefteq S_{i-1} \times S_j$ such that $\varepsilon_{i-1} \times N' \subseteq \bar{N}$;*
6. *a dlock of type $\varepsilon\mathcal{H}$ with normal subgroup $\bar{N} \trianglelefteq S_{i-1} \times S_j$ such that $\varepsilon_{i-1} \times N' \subseteq \bar{N}$.*

Dlocks of type $\varepsilon\mathcal{H}$ and $\mathcal{H}\varepsilon$

The dlocks of type $\varepsilon\mathcal{H}$ can be placed onto the "west-facing" slopes of the landscape made up of the dlocks of type FF, $\mathcal{H}F$, or εF. For any such slope the dlocks of type $\varepsilon\mathcal{H}$, must be "staked on the top of each other".



The final dlock type $\varepsilon\varepsilon$ can occur in any remaining $\mathcal{D}$-class.

Theorem

Suppose that we are given a partition $\mathcal{P}$ of $\mathcal{T}_m \times \mathcal{T}_n$ that preserves $\mathcal{D}$-classes and that to each part B of $\mathcal{P}$, we associate a type from $\{F, \mathcal{H}, \varepsilon\}^2$ and, if the type of B differs from FF, a group N_B. Suppose further that a VERY LONG LIST OF CONDITIONS hold.
On each $\mathcal{P}$-part B we define a binary relation θ_B as follows:

(i) *If B has type FF let $\theta_B = B^2$;*
(ii) *If B has type $\mathcal{H}F$ or εF, let $(f,g)\theta_B(f',g')$ if and only if $f\mathcal{H}f'$ and $f' = f\cdot\sigma$ for some $\sigma \in N_B$;*
(iii) *If B has type $F\mathcal{H}$ or $F\varepsilon$, let $(f,g)\theta_B(f',g')$ if and only if $g\mathcal{H}g'$ and $g' = g\cdot\sigma$ for some $\sigma \in N_B$;*
(iv) *If B has type $\mathcal{H}\mathcal{H}$, $\varepsilon\mathcal{H}$, $\mathcal{H}\varepsilon$, or $\varepsilon\varepsilon$, let $(f,g)\theta_B(f',g')$ if and only if $f\mathcal{H}f'$, $g\mathcal{H}g'$ and $(f',g') = (f\cdot\sigma, g\cdot\tau)$ for some $(\sigma,\tau) \in N_B$.*

Let $\theta = \cup_{B\in\mathcal{P}}\theta_B$. Then θ is a congruence on $\mathcal{T}_m \times \mathcal{T}_n$.
Conversely, every congruence on $\mathcal{T}_m \times \mathcal{T}_n$ can be obtained in this way.

- Everything we said also holds (with minute modifications) for $\mathcal{PT}_m \times \mathcal{PT}_n$ and $\mathcal{I}_m \times \mathcal{I}_n$.
- In fact, it also holds for mixed products such as $\mathcal{PT}_3 \times \mathcal{I}_7$.
- We obtained similar results for products of matrix moniods $\mathcal{M}(n, F)$ over a field F.
- These cases include an extra complication, as the ε-type properly corresponds to matrices that are scalar multiples of each other. The multiplier appears as an extra parameter.
- Finally, transferring our results to $\mathcal{T}_m \times \mathcal{T}_n \times \mathcal{T}_k$ is "straightforward", but complex.

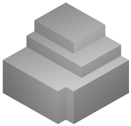


Figure: F, F, F

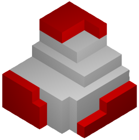


Figure: $F, F, \mathcal{H}$

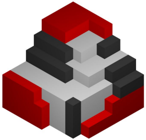


Figure: $F, \mathcal{H}, \mathcal{H}$

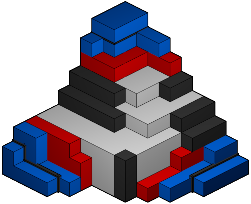


Figure: F, F, ε

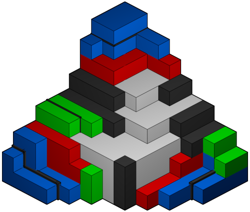


Figure: $F, \mathcal{H}, \varepsilon$

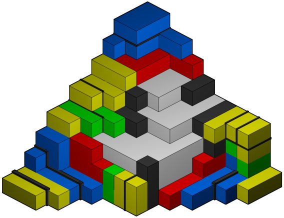


Figure: $F, \varepsilon, \varepsilon$

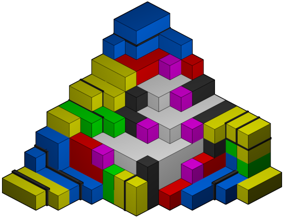


Figure: $\mathcal{H}, \mathcal{H}, \mathcal{H}$

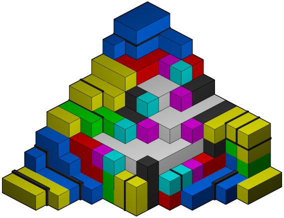


Figure: $\mathcal{H}, \mathcal{H}, \varepsilon$

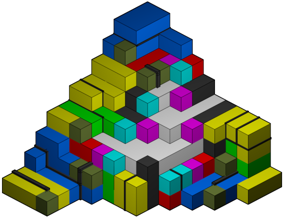


Figure: $\mathcal{H}, \varepsilon, \varepsilon$

Happy New Year!