MORITA EQUIVALENCE OF SEMIGROUPS REVISITED: FIRM SEMIGROUPS

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Monoids

Two monoids S and T are Morita equivalent if there exist elements $u, v, w \in T$ such that $u^2 = u$, $vu = v$, $uw = w$, $vw = 1$ and $uTu \cong S$.

Corollary 1.
Morita equivalence of two monoids very often reduces to isomorphism.

Corollary 2.
The categories of all acts over two arbitrary semigroups are equivalent if and only if the two semigroups are isomorphic.

Semigroups, acts 1

factorisable semigroup: every element can be written as a product
semigroup S with weak local units:

$$\forall s \in S \;\; \exists u, v \in S \;\; su = s = vs$$

semigroup S with local units: u and v can be chosen to be idempotent

A_S *unitary act*: $AS = A$
A_S *firm act*: the mapping

$$\mu_A : A \otimes S \to A, \;\; a \otimes s \mapsto as$$

is bijective

A_S unitary $\iff$ μ_A surjective, hence firm acts are unitary.
S factorisable $\iff$ μ_S surjective

Semigroups, acts 2

S is a **firm semigroup**: S_S (or ${}_SS$) is a firm act

S is a *right fair semigroup*: every subact of a unitary right S-act is unitary.
S is a *fair semigroup*: S is left and right fair.
A_S *s-unital act*: for every $a \in A$ there exists $s \in S$ such that $as = a$.
S is a right fair semigroup $\iff$ every unitary right S-act is s-unital.

Fair semigroups are counterparts of *xst-rings* considered by García and Marín, based on work of Xu, Shum, and Turner-Smith.

A_S *non-singular act*: if $as = a's$ for all $s \in S$, then $a = a'$ $(a, a' \in A)$

Categories of acts

Act_S (${}_S\mathrm{Act}, {}_S\mathrm{Act}_T$) all right S-acts (left S-acts, (S, T)-biacts)
UAct_S all unitary right S-acts
FAct_S all firm right S-acts
NAct_S all non-singular right S-acts
CAct_S those A_S acts for which the mapping

$$\lambda_S\colon S \longrightarrow \mathrm{Act}_S(S, A):\ \lambda_S(a)(s) = as$$

is an isomorphism
$\mathrm{Fx} - \mathrm{Act}_S$ all fixed acts: those acts A_S for which the mapping

$$S \otimes S\mathrm{Act}(S, A) \to A:\ s \otimes f \mapsto (s)f$$

is an isomorphism

Adjoint situations, equivalences 1

Let S be a firm semigroup.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{Act}_S \arrow[r, phantom, "="] \arrow[d, "\mathsf{Act}_S(S,-)" description] & \mathsf{Act}_S \arrow[r, bend left, "-S"] \arrow[r, phantom, "\top"] \arrow[d, "-\otimes S" description] & \mathsf{UAct}_S \arrow[l, bend left, "I"] \arrow[d, "\mathsf{Act}_S(S,-)S" description] \\
\mathsf{CAct}_S \arrow[u, bend left=60, "-\otimes S"] \arrow[u, bend right=60, "I"'] \arrow[r, bend left, "-\otimes S"] \arrow[r, phantom, "\approx"] & \mathsf{FAct}_S \arrow[l, bend left, "\mathsf{Act}_S(S,-)"] \arrow[u, bend left=60, "I"] \arrow[u, bend right=60, "\mathsf{Act}_S(S,-)"'] \arrow[r, bend left, "\mathsf{Act}(S,-)S"] \arrow[r, phantom, "\approx"] & \mathsf{NAct}_S \arrow[l, bend left, "-\otimes S"] \arrow[u, bend left=60, "-\otimes S"] \arrow[u, bend right=60, "I"']
\end{tikzcd}
$$

The equivalence between CAct_S and NAct_S can be given as

$$
\begin{tikzcd}[column sep=huge]
\mathsf{CAct}_S \arrow[r, shift left, "-S"] & \mathsf{NAct}_S. \arrow[l, shift left, "\mathsf{Act}_S(S,-)"]
\end{tikzcd}
$$

Adjoint situations, equivalences 2

FAct_S is an essential colocalization of UAct_S with coreflection $-\otimes S$;
NAct_S is an essential localization of UAct_S with reflection $\mathsf{Act}_S(S,-)S$.

Over a firm semigroup, firm acts are the same as fixed acts.

Equivalence functors between categories of firm acts

Let S and T be firm semigroups and ${}_SP_T$ be a biact such that P_T is firm. Then the functor $- \otimes P\colon \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_T$ is left adjoint to the functor $\mathsf{Act}_T(P, -) \otimes S\colon \mathsf{FAct}_T \longrightarrow \mathsf{FAct}_S$.

Let S and T be firm semigroups and $F : \mathsf{FAct}_S \to \mathsf{FAct}_T$ and $G : \mathsf{FAct}_S \to \mathsf{FAct}_T$ be mutually inverse equivalence functors. Then

$$F \cong \mathsf{Act}_S(G(T), -) \otimes T \quad \text{a} \quad G \cong \mathsf{Act}_T(F(S), -) \otimes S.$$

Let S and T be firm semigroups and $F : \mathsf{FAct}_S \to \mathsf{FAct}_T$ and $G : \mathsf{FAct}_S \to \mathsf{FAct}_T$ be mutually inverse equivalence functors. Then

$$F \cong - \otimes F(S),$$
$$G \cong - \otimes G(T).$$

Moreover, the left acts ${}_SF(S)$ and ${}_TG(T)$ are firm.

Morita contexts

a sixtuple $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, where S and T are semigroups, ${}_SP_T \in {}_S\mathrm{Act}_T$ ands ${}_TQ_S \in {}_T\mathrm{Act}_S$ are biacts, and

$$\theta : {}_S(P \otimes Q)_S \to {}_SS_S, \quad \phi : {}_T(Q \otimes P)_T \to {}_TT_T$$

are biact morphisms such that, for every $p, p' \in P$ and $q, q' \in Q$,

$$\theta(p \otimes q)p' = p\phi(q \otimes p'), \quad q\theta(p \otimes q') = \phi(q \otimes p)q'.$$

A Morita context is

- *unitary* if ${}_SP_T$ and ${}_TQ_S$ are unitary biacts,
- *surjective* if θ and ϕ are surjective,
- *bijective* if θ and ϕ are bijective.

Morita equivalence and strong Morita equivalence 1

Semigroups S and T are *(right) Morita equivalent* if the categories FAct_S and FAct_T are equivalent.
Semigroups S and T are *strongly Morita equivalent*, if they are contained in a unitary surjective Morita context.

Earlier results:

If S is strongly Morita equivalent to any semigroup, including itself, then S is factorisable.

Lawson

(Right) Morita equivalence and strong Morita equivalence coincide for semigroups with local units.

Morita equivalence and strong Morita equivalence 2

Chen–Shum

For arbitrary factorisable semigroups S és T, the categories NAct_S and NAct_T are equivalent if and only if the semigroups S/ζ_S and T/ζ_T are strongly Morita equivalent, where the congruence ζ_A is defined, for an act A_S, by

$$\zeta_A = \{(a_1, a_2) \in A^2 \mid a_1 s = a_2 s \text{ for all } s \in S\}.$$

Laan–Márki

Let S and T be fair semigroups such that $U(S)$ and $U(T)$ have common weak local units, where $U(S) = \{s \in S \mid s = us = sv \text{ for some } u, v \in S\}$ (this is an ideal in S). Then S and T are right Morita equivalent if and only if $U(S)$ and $U(T)$ are strongly Morita equivalent.

Morita equivalence need not be strong

non-factorisable example

S is a non-trivial semigroup with zero multiplication: then S is fair and $U(S) = \{0\}$ has common weak local units. S is right and left Morita equivalent to the one-element semigroup but it is not factorisable.

Main theorem

For firm semigroups S and T, the following conditions are equivalent:

1. The categories FAct_S and FAct_T are equivalent.
2. The categories ${}_S\mathrm{FAct}$ and ${}_T\mathrm{FAct}$ are equivalent.
3. There exists a unitary bijective Morita context containing S and T.
4. There exists a unitary surjective Morita context containing S and T.
5. There exists a surjective Morita context containing S and T.