# Exploring Quasi-crystals and algebraic structures: linking cyrstal bases to semigroups and beyond

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# Motivation

## Plactic monoid
[Lascoux, Schützenberger '81]

- Young tableaux, Schensted insertion

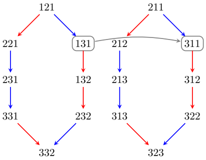
| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 2 | | |
| 3 | | | |

- Knuth relations

$$acb \equiv cab, \; a \leq b < c$$

$$bac \equiv bca, \; a < b \leq c$$

- Crystals




## Hypoplactic monoid
[Krob, Thibon '97], [Novelli '00]

- Quasi-ribbon tableaux, Krob–Thibon insertion

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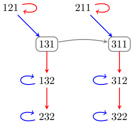
| 1 | 1 | | | |
| | 2 | 2 | 2 | |
| | | | 3 | 3 |

- Knuth + quartic relations

$$cadb \equiv acbd, \; a \leq b < c \leq d$$

$$bdac \equiv dbca, \; a < b \leq c < d$$

- Quasi-crystals



# Crystals

**Definition**

A **crystal** of type $A_{n-1}$ is a non-empty set $\mathcal{C}$ together with maps

$$\tilde{e}_i, \tilde{f}_i : \mathcal{C} \longrightarrow \mathcal{C} \sqcup \{\perp\} \qquad \text{(Kashiwara operators)}$$
$$\tilde{\varepsilon}_i, \tilde{\varphi}_i : \mathcal{C} \longrightarrow \mathbb{Z} \sqcup \{-\infty\} \qquad \text{(length functions)}$$
$$wt : \mathcal{C} \longrightarrow \mathbb{Z}^n \qquad \text{(weight function)}$$

for $i \in I := \{1, \ldots, n-1\}$, satisfying the following:

**C1.** For any $x, y \in \mathcal{C}$, $\tilde{e}_i(x) = y$ iff $x = \tilde{f}_i(y)$, and in that case

$$wt(y) = wt(x) + \alpha_i, \quad \tilde{\varepsilon}_i(y) = \tilde{\varepsilon}_i(x) - 1, \quad \tilde{\varphi}_i(y) = \tilde{\varphi}_i(x) + 1$$

**C2.** $\tilde{\varphi}_i(x) = \tilde{\varepsilon}_i(x) + \langle wt(x), \alpha_i \rangle$

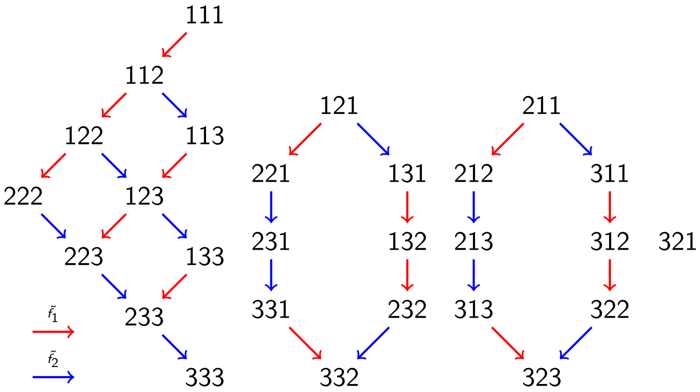
**C3.** $\tilde{\varepsilon}_i(x) = -\infty \Rightarrow \tilde{e}_i(x) = \tilde{f}_i(x) = \perp$.

where $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$.

(This definition is generalized for other Cartan types)

# Crystals

- The **crystal graph** associated to a crystal $\mathcal{C}$ is the directed weighted graph where $y \xrightarrow{i} x$ iff $\tilde{e}_i(x) = y$ iff $\tilde{f}_i(y) = x$.

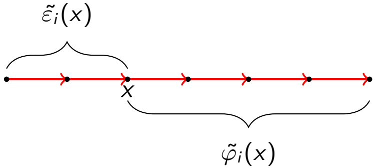


# Crystals

- A crystal is **seminormal** if

$$\tilde{\varepsilon}_i(x) = \max\{k : \tilde{e}_i(x)^k \neq \perp\}, \quad \tilde{\varphi}_i(x) = \max\{k : \tilde{f}_i(x)^k \neq \perp\},$$

  for all $i \in I$ and $x \in \mathcal{C}$. In particular, $\tilde{\varepsilon}_i(x), \tilde{\varphi}_i(x) \geq 0$.




- To compute $\tilde{f}_i(w)$ and $\tilde{e}_i(w)$ on a word $w \in \{1 < \cdots < n\}^*$:
  - consider the subword with only symbols $i$ and $i+1$, and cancel all pairs $(i+1)i$ ($i$-inversions), until there are no pairs left.
  - $\tilde{e}_i$ changes the *leftmost* $i+1$ to $i$, if possible; if not, it is $\perp$.
  - $\tilde{f}_i$ changes the *rightmost* $i$ to $i+1$, if possible; if not, it is $\perp$.

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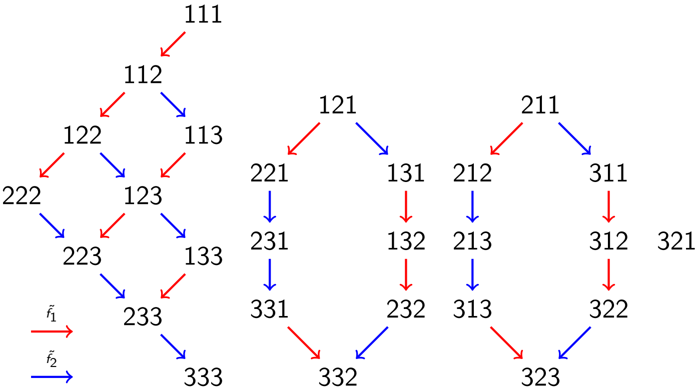
# Stembridge crystals

- A **Stembridge crystal** is a seminormal crystal of simply-laced type that satisfies some local axioms [Stembridge '03]. These are the crystal graphs that correspond to representations of Lie algebras.
- The connected components have nice properties:
  - Unique highest weight element (source vertex), from which all vertices can be reached.
  - All components whose highest weight elements have the same weight are isomorphic.
  - In type $A$, the character of a connected component is a Schur function $s_\lambda$.

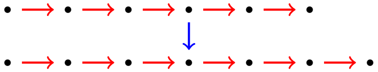
# Stembridge crystals



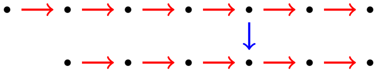
# Stembridge crystals

## Local axioms

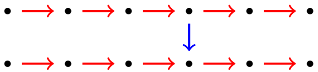
**S1.** If $\tilde{e}_i(x) = y$, then $\tilde{\varepsilon}_j(y)$ is equal to $\tilde{\varepsilon}_j(x)$ or $\tilde{\varepsilon}_j(x) + 1$ (the second case is possible only if $|i - j| = 1$).



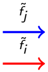
or



for $|i - j| = 1$



for $|i - j| > 1$

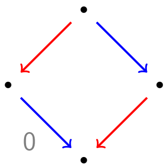


# Stembridge crystals

## Local axioms

**S2.** If $\tilde{e}_i(x) = y$ and $\tilde{\varepsilon}_j(y) = \tilde{\varepsilon}_j(x) > 0$ then

$$\tilde{e}_i\tilde{e}_j(x) = \tilde{e}_j\tilde{e}_i(x) \neq \perp$$

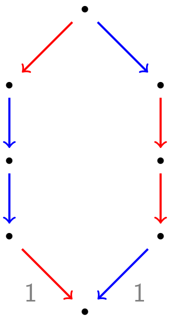



and $\tilde{\varphi}_i(\tilde{e}_j(x)) = \tilde{\varphi}_i(x)$.

(and dual axioms for $\tilde{f}_i$, $\tilde{f}_j$)

**S3.** If $\tilde{e}_i(x) = y$ and $\tilde{e}_j(x) = z$, and $\tilde{\varepsilon}_i(z) = \tilde{\varepsilon}_i(x) + 1$ and $\tilde{\varepsilon}_j(y) = \tilde{\varepsilon}_j(x) + 1$ then

$$\tilde{e}_i\tilde{e}_j{}^2\tilde{e}_i(x) = \tilde{e}_j\tilde{e}_i{}^2\tilde{e}_j(x) \neq \perp.$$



# Motivation

**Plactic monoid**
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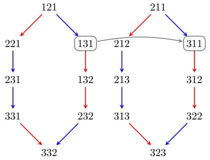
- Crystals

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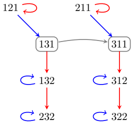
- Quasi-crystals



# Quasi-crystals

- First introduced by Cain and M. (2017), providing another characterization of the hypoplactic monoid of type $A$.
- Cain, Guilherme and M. (2023) provided a definition of abstract quasi-crystals for other Cartan types.
- For type $A$, each connected component has a unique highest weight element, is isomorphic to a quasi-crystal of quasi-ribbon tableaux, and its character is a fundamental quasisymmetric function $F_\alpha$.
- We have $s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{DesComp(T)}$.
- Using this decomposition, Maas-Gariépy (2023) independently introduced quasi-crystals, as subgraphs of a connected component of a crystal graph.

# Quasi-crystals

**Definition (Cain, Guilherme, M. '23)**

A **quasi-crystal** of type $A_{n-1}$ is a non-empty set $\mathcal{Q}$ together with maps

$$\begin{aligned} \ddot{e}_i, \ddot{f}_i &: \mathcal{Q} \longrightarrow \mathcal{Q} \sqcup \{\perp\} && \text{(quasi-Kashiwara operators)} \\ \ddot{\varepsilon}_i, \ddot{\varphi}_i &: \mathcal{Q} \longrightarrow \mathbb{Z} \sqcup \{-\infty, +\infty\} \\ wt &: \mathcal{Q} \longrightarrow \mathbb{Z}^n \end{aligned}$$

for $i \in \{1, \ldots, n-1\}$, satisfying the same axioms of crystals and additionally:

$$\ddot{\varepsilon}_i(x) = +\infty \Rightarrow \ddot{e}_i(x) = \ddot{f}_i(x) = \perp.$$

- A quasi-crystal is **seminormal** if, for all $i \in I$ and $x \in \mathcal{Q}$,

  $$\ddot{\varepsilon}_i(x) = \max\{k : \ddot{e}_i(x)^k \neq \perp\}, \quad \ddot{\varphi}_i(x) = \max\{k : \ddot{f}_i(x)^k \neq \perp\}$$

  whenever $\ddot{\varepsilon}_i(x) \neq +\infty$.

# Quasi-crystals

To compute $\ddot{f}_i(w)$ and $\ddot{e}_i(w)$ on a word $w \in \{1 < \cdots < n\}^*$:

- If $w$ has an $i$-inversion, $\ddot{f}_i(w) = \ddot{e}_i(w) = \perp$.
- Otherwise, $\ddot{f}_i(w) = \tilde{f}_i(w)$ and $\ddot{e}_i(w) = \tilde{e}_i(w)$.

$$\ddot{f}_1(13121) =$$
$$\ddot{f}_1(13112) =$$

$$\ddot{f}_1(13121) =$$
$$\ddot{f}_1(13112) =$$

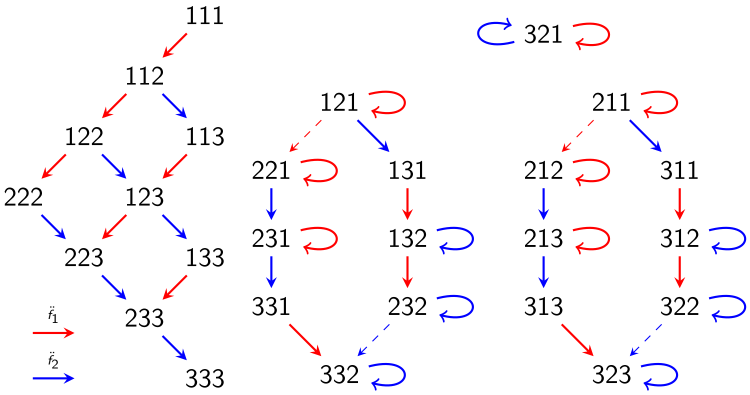
$$\ddot{f}_1(13121) =$$
$$\ddot{f}_1(13112) =$$

$$\ddot{f}_1(13121) = \perp$$
$$\ddot{f}_1(13112) =$$

# Quasi-crystals

The **quasi-crystal graph** associated to a quasi-crystal $\mathcal{Q}$ is the directed weighted graph where:

- $y \xrightarrow{i} x$ iff $\ddot{e}_i(x) = y$.
- $x$ has an $i$-labelled loop iff $\ddot{\varepsilon}_i(x) = +\infty$ iff $\ddot{\varphi}_i(x) = +\infty$.

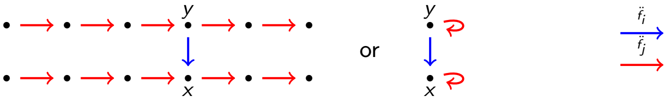


# Local characterization of quasi-crystals

## Local quasi-crystal axioms

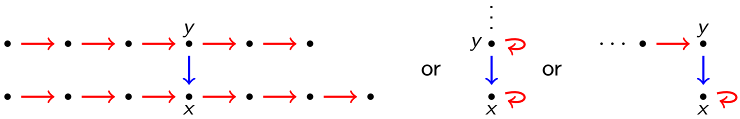
**LQC1.** If $\ddot{e}_i(x) = y$, then:

- For $|i-j| > 1$, $\ddot{\varepsilon}_j(x) = \ddot{\varepsilon}_j(y)$.



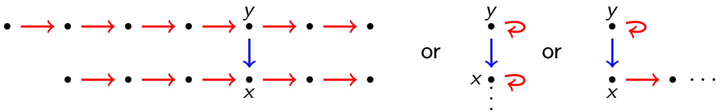

- For $j = i+1$,

$$\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y) \Leftrightarrow \left(\ddot{\varepsilon}_{i+1}(x) = +\infty \wedge \ddot{\varepsilon}_i(y) = 0\right) \Rightarrow \ddot{\varepsilon}_{i+1}(y) > 0.$$




- For $j = i-1$,

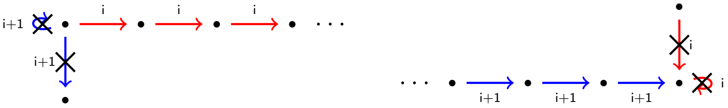
$$\ddot{\varphi}_{i-1}(x) \neq \ddot{\varphi}_{i-1}(y) \Leftrightarrow \left(\ddot{\varphi}_{i-1}(y) = +\infty \wedge \ddot{\varphi}_i(x) = 0\right) \Rightarrow \ddot{\varphi}_{i-1}(x) > 0.$$



# Local characterization of quasi-crystals

## Local quasi-crystal axioms

**LQC2.** $\ddot{\varepsilon}_i(x) = 0$ iff $\ddot{\varphi}_{i+1}(x) = 0$, for $i \in \{1, \ldots, n-2\}$.




**LQC3.** If both $\ddot{e}_i(x)$ and $\ddot{e}_j(x)$ are defined, for $i \neq j$, then
$\ddot{e}_i\ddot{e}_j(x) = \ddot{e}_j\ddot{e}_i(x) \neq \perp$ (and dual axiom for $\ddot{f}_i$, $\ddot{f}_j$.)

$$\begin{tikzcd}
 & \ddot{e}_i\ddot{e}_j(x) = \ddot{e}_j\ddot{e}_i(x) \arrow[dl, red] \arrow[dr, blue] & \\
\ddot{e}_j(x) \arrow[dr, blue] & & \ddot{e}_i(x) \arrow[dl, red] \\
 & x &
\end{tikzcd}$$

# Local characterization of quasi-crystals

**Theorem (Cain, M., Rodrigues, Rodrigues '23)**

*If $\mathcal{Q}$ is a quasi-crystal of type $A$ (not necessarily seminormal) satisfying the local axioms, and such that $\ddot{\varepsilon}_i(x) \neq +\infty$ and $\ddot{\varphi}_i(x) \neq +\infty$, for all $i \in I, x \in \mathcal{Q}$, then $\mathcal{Q}$ is a weak Stembridge crystal (i.e. not necessarily seminormal).*

**Theorem (Cain, M., Rodrigues, Rodrigues '23)**

*Let $\mathcal{Q}$ be a connected component of a seminormal quasi-crystal graph of type $A$, weighted in $\mathbb{Z}^n_{\geq 0}$, satisfying the local axioms. Then, $\mathcal{Q}$ has a unique highest weight element, whose weight is a composition.*

**Theorem (Cain, M., Rodrigues, Rodrigues '23)**

*Let $\mathcal{Q}$ and $\mathcal{Q}'$ be connected components of seminormal quasi-crystal graphs of type $A$ satisfying the local axioms, with highest weight elements $u$ and $v$. If $wt(u) = wt(v)$, then there exists a weight-preserving isomorphism between $\mathcal{Q}$ and $\mathcal{Q}'$.*

# Quasi-tensor product of quasi-crystals

Cain, Guilherme, and M. (2023) introduced a notion of quasi-tensor product of seminormal quasi-crystals, denoted $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$, which has $\mathcal{Q} \times \mathcal{Q}'$ as underlying set and maps:

- $wt(x \mathbin{\ddot{\otimes}} x') = wt(x) + wt(x')$.
- If $\ddot{\varphi}_i(x) > 0$ and $\ddot{\varepsilon}_i(x') > 0$, $\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{f}_i(x \mathbin{\ddot{\otimes}} x') = \perp$ and $\ddot{\varepsilon}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{\varphi}_i(x \mathbin{\ddot{\otimes}} x') = +\infty$, otherwise,

$$\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = \begin{cases} \ddot{e}_i(x) \mathbin{\ddot{\otimes}} x' & \text{if } \ddot{\varphi}_i(x) \geq \ddot{\varepsilon}_i(x') \\ x \mathbin{\ddot{\otimes}} \ddot{e}_i(x') & \text{if } \ddot{\varphi}_i(x) < \ddot{\varepsilon}_i(x') \end{cases}$$

$$\ddot{f}_i(x \mathbin{\ddot{\otimes}} x') = \begin{cases} \ddot{f}_i(x) \mathbin{\ddot{\otimes}} x' & \text{if } \ddot{\varphi}_i(x) > \ddot{\varepsilon}_i(x') \\ x \mathbin{\ddot{\otimes}} \ddot{f}_i(x') & \text{if } \ddot{\varphi}_i(x) \leq \ddot{\varepsilon}_i(x') \end{cases}$$

$$\ddot{\varepsilon}_i(x) = \max\{\ddot{\varepsilon}_i(x), \ddot{\varepsilon}_i(x') - \langle wt(x), \alpha_i \rangle\}$$
$$\ddot{\varphi}_i(x) = \max\{\ddot{\varphi}_i(x) + \langle wt(x'), \alpha_i \rangle, \ddot{\varphi}_i(x')\}$$

(With this convention $x \mathbin{\ddot{\otimes}} y$ is identified with the word $yx$.)

# Quasi-tensor product of quasi-crystals

- $\mathcal{B}_n$ is the standard crystal of type $A_{n-1}$:

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \quad \cdots \quad \xrightarrow{n-1} n$$

- Similarly to the case of the plactic monoid, each component of the hypoplactic monoid is isomorphic to some $\mathcal{B}_n^{\ddot{\otimes}\, k}$.

$$\begin{tikzcd}
1 \ddot{\otimes} 2 \arrow[loop left] \arrow[dr] & \\
& 1 \ddot{\otimes} 3 \arrow[dl] \\
2 \ddot{\otimes} 3 \arrow[loop left] &
\end{tikzcd}$$

A connected component of $\mathcal{B}_3 \ddot{\otimes} \mathcal{B}_3$

$$\begin{tikzcd}
1 \otimes 2 \arrow[dr] & \\
& 1 \otimes 3 \arrow[dl] \\
2 \otimes 3 &
\end{tikzcd}$$

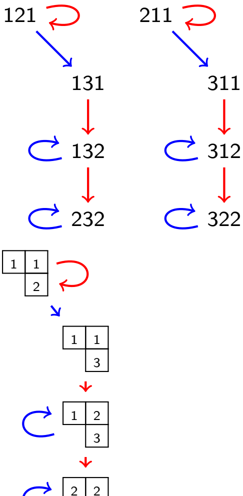
A connected component of $\mathcal{B}_3 \otimes \mathcal{B}_3$

# Quasi-tensor product of quasi-crystals

**Theorem (Cain, M., Rodrigues, Rodrigues '23)**

*Let $\mathcal{Q}$ and $\mathcal{Q}'$ be seminormal quasi-crystal graphs satisfying the local axioms. Then, $\mathcal{Q} \ddot{\otimes} \mathcal{Q}'$ is a seminormal quasi-crystal that satisfies the same axioms.*

- The standard crystal $\mathcal{B}_n$ satisfies the local axioms.
- A connected component of a quasi-crystal of words, being isomorphic to some $\mathcal{B}_n^{\ddot{\otimes} k}$, satisfies the local axioms.
- As a consequence, every connected component of a seminormal quasi-crystal satisfying the local axioms is isomorphic a quasi-crystal of quasi-ribbon tableaux.



# From crystals to quasi-crystals

Let $(\mathcal{C}, \tilde{f}_i, \tilde{e}_i, \tilde{\varepsilon}_i, \tilde{\varphi}_i)$ be a connected component of a Stembridge crystal, weighted in $\mathbb{Z}^n_{\geq 0}$, and define $(\mathcal{Q}, \ddot{f}_i, \ddot{e}_i, \ddot{\varepsilon}_i, \ddot{\varphi}_i)$ to have the same underlying set as $\mathcal{C}$ and:

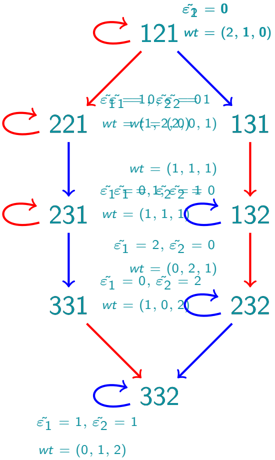
- Place a $i$-labelled loop on $x$ if $\tilde{\varepsilon}_i(x) < wt_{i+1}(x)$, for all $i \in I, x \in \mathcal{C}$ (equivalently, if $\tilde{\varphi}_i(x) < wt_i(x)$).
- Then, remove $i$-labelled edges that have $i$-labelled loops on both ends.

**Theorem (Cain, M., Rodrigues, Rodrigues '23)**

*$\mathcal{Q}$ is a seminormal quasi-crystal that satisfies the local axioms.*

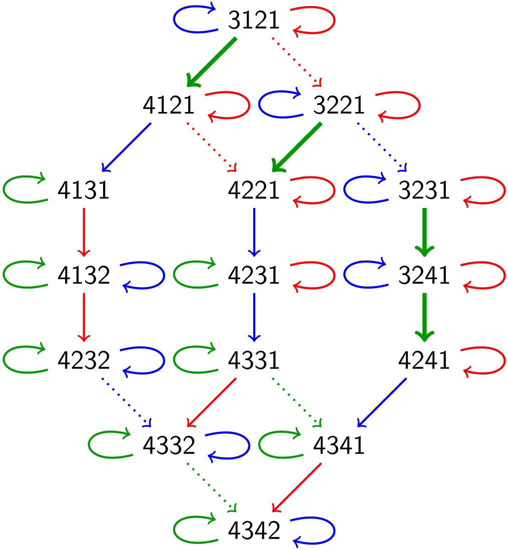
**Corollary (Cain, M., Rodrigues, Rodrigues '23)**

*If $\mathcal{C}$ has highest weight $\lambda$, the number of connected components of $\mathcal{Q}$ is given by the number of standard Young tableaux of shape $\lambda$.*




$s_{21} = F_{21} + F_{12}.$

# From crystals to quasi-crystals




$$s_{211} = F_{211} + F_{121} + F_{112}$$

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Thank you!