# The algebra of the monoid of order-preserving functions and other reduced E-Fountain semigroups

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# The monoid of order-preserving functions

- A function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is order-preserving if $i \leq j \implies f(i) \leq f(j)$.
- We denote by $\mathcal{O}_n$ the monoid of all order-preserving functions on $\{1, \ldots, n\}$.

## Example

$$f = \left(\begin{array}{cc|cc|cccc|c} 1 & \color{red}{2} & 3 & \color{red}{4} & 5 & 6 & 7 & \color{red}{8} & \color{red}{9} \\ 2 & 2 & 5 & 5 & 7 & 7 & 7 & 7 & 8 \end{array}\right) \in \mathcal{O}_9$$

$$X = \{2, 4, 8, 9\}, \quad Y = \{2, 5, 7, 8\}$$

$$f = f_{Y,X}$$

- Every $f \in \mathcal{O}_n$ corresponds to $X, Y \subseteq \{1, \ldots, n\}$ with $|X| = |Y|$ and $n \in X$. We set $f = f_{Y,X}$.

# Associated "category"

- Goal: We want to find a ~~category~~ category-like structure $C_n$ such that $\Bbbk\,\mathcal{O}_n \simeq \Bbbk C_n$ for any commutative unital ring $\Bbbk$. This approach is useful in the study of many semigroup algebras.
- A semigroup algebra

$$\Bbbk S = \{\sum k_i s_i \mid k_i \in \Bbbk \quad s_i \in S\}$$

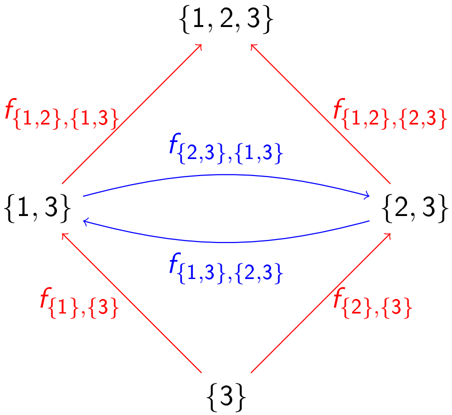
- For a "category-like" structure $C$, an algebra $\Bbbk C$ is the free $\Bbbk$-module of all linear combinations

$$\{\sum k_i m_i \mid k_i \in \Bbbk,\ m_i \text{ morphism}\}$$

  - if $\nexists m_i \cdot m_j$ then $m_i \cdot m_j = 0$ in $\Bbbk C$.

# Associated "category" of $\mathcal{O}_n$

- The objects of $C_n$: Subsets $X \subseteq \{1, \ldots, n\}$ with $n \in X$.
- Morphisms: With every $f_{Y,X} \in \mathcal{O}_n$ we associate a morphism from $X$ to $Y \cup \{n\}$.
- Example: $C_n$ for $n = 3$.



# Associated "category" of $\mathcal{O}_n$

- Composition in $C_n$ is defined as follows:

$$f_{W,Z} \bullet f_{Y,X} = \begin{cases} f_{W,Z} f_{Y,X} & \text{The domain of } f_{W,Z} \text{ is the range of } f_{Y,X} \\ & (Z = Y \cup \{n\}) \\ & \text{and } (n \in W \vee n \in Y) \\ \text{undefined} & \text{otherwise} \end{cases}$$

- $C_n$ is a category with partial composition. If $m_3, m_2, m_1$ are three morphisms then

$$\exists m_1 \cdot (m_2 \cdot m_3) \iff \exists (m_1 \cdot m_2) \cdot m_3$$

and in this case $m_1 \cdot (m_2 \cdot m_3) = (m_1 \cdot m_2) \cdot m_3$.

# Associated "category" of $\mathcal{O}_n$

## Proposition (IS)

*For every commutative unital ring* $\Bbbk$, $\Bbbk\,\mathcal{O}_n \simeq \Bbbk C_n$.

## Remark

Morita equivalence between $\Bbbk\,\mathcal{O}_n$ and $\Bbbk C_n$ follows from the Dold-Kan correspondence.

# A generalization

We want to obtain a "$\Bbbk S \simeq \Bbbk C$ Theorem" that generalizes the following cases:

- $\mathcal{O}_n$
- Reduced $E$-Fountain semigroups with the congruence condition (IS 2022) - This case includes inverse semigroups, monoids of partial functions and the Catalan monoid (Steinberg, Margolis, IS).
- Strict right ample semigroups (Guo & Guo 2018).
- $S^2 = S$ with central idempotents (Steinberg 2022).

# Reduced $E$-Fountain semigroups

- Let $S$ be a semigroup and let $E \subseteq S$ be a subset of idempotents such that $ef = e \iff fe = e \quad \forall e, f \in E$.
- Assume that every $a \in S$ has a minimum right identity from $E$ denoted $a^*$.
  - If $e \in E$ then $ae = a \implies a^* \leq e \quad (\iff a^* e = a^* = ea^*)$.
- Assume that every $a \in S$ has a minimum left identity from $E$ denoted $a^+$.
  - If $e \in E$ then $ea = a \implies a^+ \leq e \quad (\iff a^+ e = a^+ = ea^+)$.
- In this case $S$ is called a reduced $E$-Fountain semigroup (aka DR-semigroup).

# Reduced $E$-Fountain streets?

# Reduced $E$-Fountain semigroups-examples

- Every inverse semigroup $S$ is reduced $E$-Fountain semigroup with $E = E(S)$.
  - $a^{*} = a^{-1}a, \quad a^{+} = aa^{-1}$.
- The monoid $\mathcal{O}_n$ is reduced $E$-Fountain with $E = \{f_{X,X} \mid n \in X\}$.
  - $(f_{Y,X})^{*} = f_{X,X}, \quad (f_{Y,X})^{+} = f_{Y\cup\{n\},Y\cup\{n\}}$

# Associated category??

- We can define a graph $C(S)$ as follows:
  - objects: The set $E$.
  - Morphisms: For every $a \in S$ we associated a morphism $C(a)$ from $a^*$ to $a^+$.
- Naive definition of composition:

$$C(b) \bullet C(a) = \begin{cases} C(ba) & b^* = a^+ \\ \text{undefined} & \text{otherwise} \end{cases}$$

- Problem: There is no reason that $b^* = a^+ \implies (ba)^* = a^* \wedge (ba)^+ = b^+$. The customary way to avoid this problem is to assume the congruence conditions

$$(ab)^* = (a^* b)^* \quad (ab)^+ = (ab^+)^+$$

-but we cannot do so!

# Associated category with partial composition??

- Another attempt of defining composition:

$$C(b) \bullet C(a) = \begin{cases} C(ba) & b^* = a^+, \quad (ba)^* = a^*, \quad (ba)^+ = b^+. \\ \text{undefined} & \text{otherwise} \end{cases}$$

- <u>Problem</u>: There is no reason that this will be partially associative

$$\exists m_1 \cdot (m_2 \cdot m_3) \iff \exists (m_1 \cdot m_2) \cdot m_3$$

# Generalized right ample identity

- A reduced $E$-Fountain semigroup $S$ satisfies the generalized right ample identity if one of the following equivalent conditions holds:
  - $\forall a, b \in S \qquad (b(a(ba)^*)^+)^* = (a(ba)^*)^+.$
  - $\forall a, b \in S \qquad (ba)^* = a^* \implies (ba^+)^* = a^+.$
- The monoid $\mathcal{O}_n$ satisfies the generalized ***left*** ample identity.

## Proposition (IS)

*Let $S$ be a reduced $E$-Fountain semigroup which satisfies the generalized right ample identity, then the graph $C(S)$ with (partial) composition defined by*

$$C(b) \bullet C(a) = \begin{cases} C(ba) & b^* = a^+, \quad (ba)^* = a^*, \quad (ba)^+ = b^+. \\ \text{undefined} & \text{otherwise} \end{cases}$$

*is a category with partial composition.*

# Goal Achieved

**Theorem (IS)**

*If $S$ is a finite reduced $E$-Fountain + generalized right ample identity + another technical condition. Then*

$$\Bbbk S \simeq \Bbbk \mathcal{C}(S)$$

*where $\mathcal{C}(S)$ is the associated category with partial composition and $\Bbbk$ is any unital commutative ring.*

# Is there any time left? .... another example

- Consider the monoid $\mathsf{B}_n$ of all binary relations with "angelic" composition:

$$\beta \cdot \alpha = \{(x, y) \mid \exists z \quad (x, z) \in \alpha, \quad (z, y) \in \beta\}$$

- Let $\mathsf{B}_n^d$ be the monoid with the same underlying set with "demonic" composition:

$$\beta * \alpha = \{(x, y) \in \beta \cdot \alpha \mid (x, z) \in \alpha \implies z \in \mathrm{dom}(\beta)\}.$$

  This is a reduced $E$-Fountain semigroup which satisfies the right congruence condition and the right (generalized) ample condition.

## Corollary

*Let $\mathcal{C}$ be the category whose objects are subsets of $\{1, \ldots, n\}$ and the hom-set $\mathcal{C}(X, Y)$ (for $X, Y \subseteq \{1, \ldots, n\}$) contains all total onto relations from $X$ to $Y$. Then*

$$\Bbbk \mathsf{B}_n^d \simeq \Bbbk\mathcal{C}$$

*for every commutative unital ring $\Bbbk$.*

Thank you!