Two quite different semigroup problems

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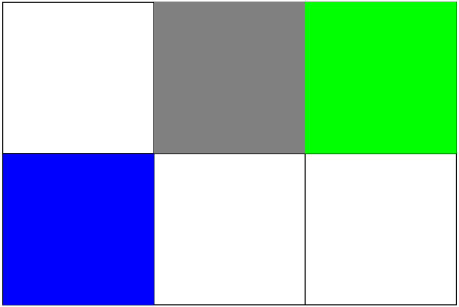
1. Matching of a finite regular semigroup S ...

... is a permutation ϕ of S such that for all $a \in S$ $(a, a\phi) \in V(S)$.

Examples

1. S an inverse semigroup, $a \mapsto a^{-1}$ is the unique matching of S.
2. Any union of groups also has an involution matching $a \mapsto a^{-1}$, where a^{-1} is the group inverse of a in H_a.
3. S is a rectangular band iff every permutation is a matching.
4. The identity mapping on S is a matching iff S satisfies $x = x^3$.

2: **7-element orthodox semigroup with no matching**



3. When does a matching exist?

Theorem

([1] Prop. 1.2 & Theorem 1.6)

For a finite regular semigroup S the following are equivalent:

(i) S has a permutation matching;

(ii) $|A| \leq |V(A)|$ for all $A \subseteq S$;

(iii) S has a permutation matching that preserves the $\mathcal{H}$-relation;

(iv) each principal factor D_a has a permutation matching;

(v) each 0-rectangular band $D_a/\mathcal{H}$ has a permutation matching.

4. Not closed under morphic images and subsemigroups

The class $\mathcal{C}$ of finite regular semigroups with a (permutation) matching is closed under the taking of finite direct products, but not under the taking of retracts.

The (finite) full transformation semigroup T_n is in $\mathcal{C}$.

5. Orthodox semigroups

Definition

Let U_1 and U_2 be finite rectangular bands, let m_i (resp. n_i) be the respective number of $\mathcal{R}$-classes and $\mathcal{L}$-classes of U_i $(i = 1, 2)$.
Then U_1 and U_2 are *similar* if

$$\frac{m_1}{n_1} = \frac{m_2}{n_2}.$$

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Theorem

([2], Theorem 3.7)

Let S be a finite orthodox semigroup. Then S has a permutation matching if and only if for each 0*-rectangular band $(D_a \cup \{0\})/\mathcal{H}$, the maximal rectangular subbands are pairwise similar.*

In this case, S then possesses an involution matching.

7. E-solid semigroups

Definition

A regular semigroup S is E-solid if S satisfies the condition that for all idempotents $e, f, g \in E(S)$

$$e\mathcal{L}f\mathcal{R}g \rightarrow \exists h \in E(S) : e\mathcal{R}h\mathcal{L}g.$$

An alternative characterisation of an E-solid semigroup is a regular semigroup S for which the idempotent-generated subsemigroup of S is a union of groups.

7. Most general result: E-solid semigroups

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An alternative characterisation of an E-solid semigroup is a regular semigroup S for which the idempotent-generated subsemigroup of S is a union of groups.

Theorem

([3] Theorem 1.3.5) Let S be a finite E-solid semigroup. Then S has a permutation matching if and only if the maximal rectangular subbands of each of the 0-rectangular bands $(D_a \cup \{0\})/\mathcal{H}$ are pairwise similar.

Moreover if S has a permutation matching then S has an involution matching.

8. Graph representations

The *graph of inverses*, $G = G(S, E)$ has an edge uv if $(u, v) \in V(S)$.

A $(1, 2)$-factor is a subgraph $G' = (S, E')$ consisting of a set of disjoint edges and cycles.

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And so S has a permutation matching if and only if G has a $(1, 2)$-factor.

Let $G = (V, E)$ be a graph (with loops) and let V' be a disjoint copy of V. Let G' be the bipartite graph with independent sets V and V' with $uv' \in E(G')$ if and only if $uv \in E(G)$.

G' is the *bipartite double-cover of* G.

Then G has a $1, 2$-factor if and only if G' has a 1-factor, (i.e. a *perfect matching*).

9. Transformation semigroup matchings

The semigroup OP_n of all orientation-preserving mappings (those that respect the orientation of the n-cycle) has a natural involution matching.

A member α of the $\mathcal{D}$-class of rank t is defined by a triple (K, R, i) where K and R are ordered t-subsets of the base set defining the initial members of the kernel classes and the range of α respectively, and i represents the 'twist' of the mapping.

The canonical inverse of α' of α is then defined by $(R, K, t - i)$.

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Open Question 1.

Does the semigroup O_n of all order-preserving mappings on the n-chain have a permutation matching?

10. The full transformation semigroup, T_n...

... has a permutation matching.

Definition

For $\alpha \in T_n$ write $\mathrm{Ker}(\alpha) = \{K_1, K_2, \cdots, K_k\}$, where the kernel classes of α are listed in ascending order of cardinality. Let $P_\alpha = (p_i)_{1 \leq i \leq k}$, where $p_i = |K_i|$. We shall say that $\alpha \mathcal{Q} \beta$ if $P_\alpha = P_\beta$. We shall write Q_α for the $\mathcal{Q}$-class of α.

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Let G be the bipartite double cover of the graph of inverses of T_n, restricted to some $\mathcal{Q}$-class Q.

Then G is *regular* (each vertex has the same positive degree), and so it follows that G has a perfect matching, and so T_n has a permutation matching.

11. Permutation matching implies involution matching?

Theorem

([7] B. Schein) T_n is covered by its inverse subsemigroups.

Theorem

([4], part of Theorem 4.1.1). If finite S possesses a closed inverse cover $\mathcal{A}$ (meaning that every non-empty intersection of a pair of members of $\mathcal{A}$ is a regular (and hence inverse) subsemigroup), then S has an involution matching.

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T_n has no closed inverse cover for $n \geq 4$.

Open Question 2. Does T_n have an involution matching?

Open Question 3. Does every semigroup with a permutation matching have an involution matching?

12. Minimal counterexample

Theorem

([4], Proposition 2.1 ff)
(a) Suppose that S is a finite regular semigroup of minimum cardinality with the property that S possesses a permutation matching but no involution matching. Then S is a 0-rectangular band.

(b) Suppose S is such a counterexample with non-zero $m \times n$ $\mathcal{D}$-class D. (wlog, $m \leq n$.) Then m is not a factor of n.

13. Colour alignment problem

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CAP: A set of m girls have mn balls so that each girl has n balls. There are m balls of each of n colours. Two girls may exchange the balls (1 ball for 1 ball) *but no ball may participate in more than one exchange*.

The goal is to achieve a situation where each girl has balls of all n colours (and so exactly one of each colour).

14. Length of short words in unavoidable sets

A set of words X over a finite alphabet A is *unavoidable* if all but finitely many words in the free monoid A^* have a factor in X.

Examples

1. $A = \{a, b\}$, $X = \{a^2, b^2, ab\}$ is unavoidable.
2. $X = A^m$.

Any unavoidable set X of size n has a word of length less that n.

Denote by M_n the maximum possible length of the shortest word taken over all unavoidable sets of size no more than n.

15. Bounds on M_n

Theorem

([6], Theorem 5, also see [5])

$$\lfloor \log_k n \rfloor \leq M_n \leq \lceil \log_k n + \log_k(\log_k n) \rceil .$$

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$$\lfloor \log_k n \rfloor \leq M_n \leq \lceil \log_k n + \log_k(\log_k n) \rceil.$$

Proof.

Define m by

$$k^m \leq n < k^{m+1} \Rightarrow m \leq \log_k n < m + 1,$$

and since m is an integer it follows that $m = \lfloor \log_k n \rfloor$. Now since $X = A^m$ is an unavoidable set all of whose members have length m and $|X| = k^m \leq n$, the left hand inequality is established.

□

15. Bounds on M_n

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$$\lfloor \log_k n \rfloor \leq M_n \leq \lceil \log_k n + \log_k(\log_k n) \rceil.$$

Let m denote the minimum length of words of X, an unavoidable set of length n. For each $w \in X$, select an m-factor $w \in A^m$, thus forming an unavoidable set X' with $|X'| \leq |X| \leq n$.

For any $v \in A^m$, X' meets all sufficiently high powers v^r, so X' contains a cyclic conjugate of v. The size of that class is at most m so that $n \geq |X| \geq |X'| \geq \frac{k^m}{m}$. Taking logarithms then gives:

$$m \leq \log_k n + \log_k m. \tag{1}$$

Now $m < n$: replace m by n on the RHS in (1) to obtain $m < 2\log_k n$. In turn we replace m by $2\log_k n$ in (1) to obtain:

$$m < \log_k n + \log_k(2\log_k n) \Rightarrow m < 1 + \log_k n + \log_k(\log_k n) \tag{2}$$

We conclude that $m \leq \lceil \log_k n + \log_k(\log_k n) \rceil$.

16. Bounds on $M_n(2)$, the second-shortest word

By considering *degenerate* sets, X (ones containing a one-letter word) we can, for $k \geq 3$, infer bounds for $M_n(2)$ from the bounds for $M_n(1)$ on a $k-1$-letter alphabet:

Theorem

([6], Theorem 2.3) Suppose that $|A| = k \geq 3$. Then

$$\lfloor \log_{k-1}(n-1) \rfloor \leq M_n(2) \leq \lceil \log_{k-1}(n-1) + \log_{k-1} \log_{k-1}(n-1) \rceil.$$

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So consider the $k = 2, A = \{a, b\}$ case. For $n = 3$ the only unavoidable sets have the form:

$$X_{l,m} = \{a^l, b^m, ab\},$$

showing that $M(2)$ does not exist for $n = 3$. Hence we restrict to minimal unavoidable sets X_n where $k = 2, n \geq 4$.

17. Codes and free monoids

A *code* C is a subset of a free monoid A^* that generates a free sub-monoid: each word in A^* has at most one factorization in C^*.

To obtain an upper bound on $M(2)$, we consider two codes, the choice depending on the nature of a shortest word $s \in X$:

$$C_1 = \{a^2, b^2\}, C_2 = \{ab, ab^2, \cdots, ab^m\},$$

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where m is the length of a second-shortest word of X.

It is C_2 that determines the upper bound for $M_n(2)$.
By considering the cyclic conjugacy class of words in $C_2^* \cap A^m$, we find that

$$|X| \geq \frac{f_{m-1}}{m},$$

where f_m denotes the mth Fibonacci number.

18. Lower bound of $M_n(2)$

The set C_m used here comprises the words of the form:

$$w = bab^{i_1}ab^{i_2}a\cdots ab^{i_j-1},$$

where $j \geq 1$, each $i_t \geq 1$ and

$$(1+i_1)+(1+i_2)+\cdots+(1+i_j) = m \geq 2.$$

Then $X = \{a^2, b^m\} \cup C_m$ is unavoidable and has order $2 + f_{m-1}$.

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Then $X = \{a^2, b^m\} \cup C_m$ is unavoidable and has order $2 + f_{m-1}$.
This leads to:
Theorem ([6], Theorem 3.2) For $n \geq 4$, taken over all minimal unavoidable sets X over $A = \{a, b\}$,

$$\lceil \log_\tau n \rceil \leq M_n(2) \leq \log_\tau n + \log_\tau(\log_\tau n) + O(1),$$

where τ denotes the golden ratio.

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