On Context-Free Inverse Graphs and Their Associated Groups

Emanuele Rodaro

Department of Mathematics
Politecnico di Milano

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Languages

Let A be a finite set, called the alphabet.

We denote by A^* the collection of all words on A.

A **language** $\mathcal{L}$ is a subset A^*.

Manageable when there is a way to produce it:

Grammar

V finite set of *non-terminal symbols*;
$S \in V$ the *start symbol*;
A finite set (disjoint from V) of *terminal symbols*;
P set of *production rules*

$$(V \cup A)^* V (V \cup A)^* \to (V \cup A)^*.$$

$$A = \{a, b\} \qquad V = \{S, L, M\}$$
$$S \to aL \mid L \to aM \mid M \to bM \mid M \to \varepsilon$$

Chomsky Hierarchy

Regular $\subset$ Context-free $\subset$ Context-sensitive $\subset$ Recursiv. enum.

$L \to a \mid L \to aM$

$L \to w(A, V) \in (A \cup V)^*$

Goal: use this classification for groups.

Language of a group

To a finitely generated group $G = \langle A \mid \mathcal{R} \rangle$ we associated the usual coding map

$$\pi : A^* \twoheadrightarrow G$$

where A is symmetric: $a \in A \Rightarrow a^{-1} \in A$.

Associated to G we have the so-called co-**word-problem language**

$$coWP(G, A) := \{w \in A^* \mid \pi(w) = \neq 1_G\}.$$

We say that a group is co-$\mathcal{C}$ if a co-word problem is $\mathcal{C}$ as a language.

Definition

We will mainly be interested in the following classes:

CF=Context-free

co-CF=Co-Context-free

poly-CF=poly-context-free (intersection of finitely many context-free languages)

Some classification results

Theorem (Anisimov, 1971)

A group is regular if and only if it is finite.

Theorem (Muller and Schupp, 1983)

The following are equivalent:

- *G is context-free;*
- *G is virtually free;*
- *the Cayley graph $Cay(G; A)$ is quasi-isometric to a tree.*

"metric comparable up to scaling factors and bounded distortion to that of a tree"

Conjectures on **poly-CF** and **co-CF** groups (I)

Theorem (Holt, Rees, Rover, Thomas (2005); Brough (2011))

poly-CF *and* **co-CF** *closed by taking direct-product, f.g. subgroups, f.i. overgroups, contains all free and abelian groups.*

Conjecture (Holt, Rees, Rover, Thomas, Brough)

Are **poly-CF** and **co-CF** groups closed under free product? It is conjectured that $\mathbb{Z} * \mathbb{Z}^2$ is neither **poly-CF**, nor **co-CF**.

Theorem (Al Kohli (2024))

$\mathbb{Z} * \mathbb{Z}^2$ *is* **co-ET0L**.

Conjecture (Ciobanu)

Grigorchuk's group is neither **poly-CF**, nor **co-CF**. In general, no **co-CF** group is torsion.

Conjecture (Brough)

Conjectures on **poly-CF** and **co-CF** groups (II)

Is there a good structural conjecture for **co-CF** groups?

Theorem (Lehnert, Schweitzer (2005))

Thompson group $\mathbb{V}$ *is in* **co-CF**

Conjecture (Lehnert)

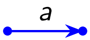
If G is **co-CF** then G embeds into Thomson $\mathbb{V}$.

Since $\mathbb{Z} * \mathbb{Z}^2$ does not embed into $\mathbb{V}$ (Bleak Salazar-Diaz 2013), if $\mathbb{Z} * \mathbb{Z}^2$ were **co-CF**, then Lehnert's conjecture would be false.

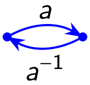
Inverse graphs

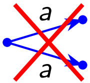
- **Directed Oriented edges:** E

- **Labeled:** A symmetric, $\mathcal{E} : E \to A$



- **Involutive:** every $e \in E$ has an opposite
  $+$ labels preserved





- **Deterministic** at most one for each label

Inverse graph $=$ all of the above $+$ connected
Examples are: Cayley graphsSchreier graphsSchützenberger graphs$=$“Cayley graphs” for inverse monoidsOthers...

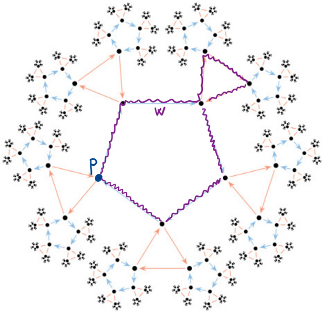
From groups to inverse graphs

Definition

An inverse graph Γ on the symmetric alphabet A is called context-free if the language of the closed walks on some root p is a context-free language:

$$L(\Gamma, p) = \{w \in A^* : p \xrightarrow{w} p\} \text{ is a context-free language}$$

- For a Cayley graph $L(\Gamma, p) = WP(G; A)$, so this definition extends the group case.



Equivalent conditions

Theorem (R. (Ceccherini-Silberstein and Woess for the complete inverse graphs case))

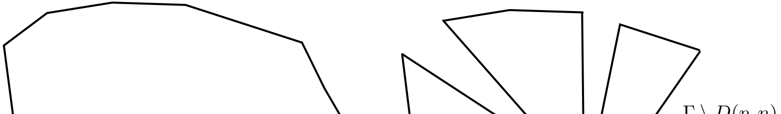
For an inverse graph Γ on the symmetric alphabet A, the following conditions are equivalent:

- Γ *is context-free;*
- Γ *is context-free as a graph (Muller-Schupp definition);*
- $\vdots$ *other conditions*

Erasing from Γ a disk $D(p, n)$ centered at p of radius n we obtain some connected components called end-cones.

Definition

Γ is a context-free graph in the sense of Muller-Schupp if there are finitely many end-cones up to end-isomorphism: an isomorphism of labeled digraphs ψ preserving the frontier points.



Generalizations of Muller-Schupp' theorem to inverse semigroups

Theorem (R.)

An inverse graph that is context-free is quasi-isometric to a tree.

Theorem (Gray, Silva, Szakacs)

Let M be a finitely presented $\langle A \mid R \rangle$ inverse monoid. If the Schützenberger graph of a word $w \in (A \cup A^{-1})^*$ is quasi-isometric to a tree, then it is context-free.

Corollary

Let M be a finitely presented $\langle A \mid R \rangle$ inverse monoid. Then the Schützenberger graph of a word $w \in (A \cup A^{-1})^*$ is context-free if and only if it is quasi-isometric to a tree.

Open problem

What happens if one substitutes finitely-presented with "regular" (or "context-free") presentation? (encoded as $u = v \to u\$\overleftarrow{v}$)

Generalizations of Muller-Schupp' theorem (II)

Theorem (R.)

For a rooted inverse graph (Γ, x_0) that is quasi-transitive (finite number of $Aut(\Gamma)$-orbits). T.F.A.E.

1. *Γ is context-free;*
2. *Γ is quasi-isometric to a tree;*
3. *$Aut(\Gamma)$ is virtually-free;*
4. *there is a covering $\psi : \Gamma \to \Lambda$ onto a finite inverse graph Λ and an homomorphism*

$$\eta : \pi_1(\Lambda, \psi(x_0)) \to \mathbb{F}_n$$

   of free groups of finite rank such that $\ker(\eta) = \overline{L(\Gamma, x_0)}$.

- It generalizes Muller-Schupp' theorem since a Cayley graph of a group is transitive.
- Fundamental for Brough's conjecture on **poly-CF** groups (similar to Schützenberger's representation theorem for **CF**)

A weak version of Brough's conjecture

Conjecture (Brough)

A group G is **poly-CF** if and only if it is virtually a finitely generated subgroup of the direct product $\mathbb{F}_{n_1} \times \ldots \times \mathbb{F}_{n_\ell}$ of finitely many free groups of finite rank.

Theorem (R.)

A group G is virtually a f.g. subgroup of $\mathbb{F}_{n_1} \times \ldots \times \mathbb{F}_{n_k}$ if and only if $WP(G; A) = \bigcap_{i=1}^{k} L(\Gamma_i, x_i)$ where Γ_i are inverse graphs that are quasi-transitive and quasi-isometric to a tree (equivalently context-free graphs).

Open problems in this direction

Open problem

As an intermediate step, what if $WP(G;A) = \bigcap_{i=1}^{k} L(\Gamma_i, x_i)$, where Γ_i are just inverse graphs that are context-free?

Note that the Cayley graph of G (transitive) is a cover of each inverse graph Γ_i.

Open problem

Is it true that if $WP(G;A) = \bigcap_{i=1}^{k} L(\Gamma_i, x_i)$, where Γ_i are just inverse graphs that are context-free, then Γ_i are quasi-transitive?

co-CF groups from inverse graphs: Transition groups

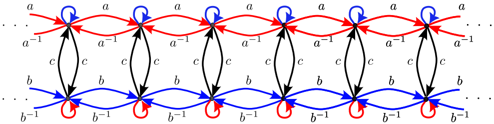
An inverse graph Γ is **complete** when for every vertex x and every $a \in A$ there exists an edge $x \xrightarrow{a} y$.

In this way, any $a \in A$ induces a permutation σ_a on the vertices. The **transition group** of Γ is

$$G(\Gamma) := \langle \sigma_a \mid a \in A \rangle.$$

Examples:

1. The transition group of a Cayley graph is the group itself.
2. $G(\;)$ is isomorphic to $\mathbb{Z}^2 \rtimes C_2$. Where is



Definition

A group is **CF-TR** if it is the transition group of disjoint union of finitely many complete **context-free** graphs (over the same alphabet).

Transition groups: some structural properties (I)

Theorem (D'Angeli, Matucci, Perego, R.)

- **CF-TR** *is a subclass of* **co-CF***.*
- *A group G belongs to* **CF-TR** *with respect to a connected graph if and only if it has a core-free subgroup H whose Schreier graph is a context-free. (core-free=trivial normal subgroup in H)*
- **CF-TR** *is closed by taking f.g. subgroups, direct products and finite index overgroups. In particular, groups that are virtually subgroups of the direct products of free groups are* **CF-TR**
- *They are never torsion (unless it is finite), in particular, Grigorchuk's group is not* **CF-TR***.*
- *It is possible to give a bound (depending on the length of a group element) on the order of an element, and so checking if an element has torsion is decidable.*

Transition groups: some structural properties (II)

Theorem (D'Angeli, Matucci, Perego, R.)

The transition group of a collection of context-free complete graphs is rational, that is, a subgroup of the group of all the homeomorphisms of the Cantor set defined by asynchronous transducers (defined by Grigorchuk, Nekrashevych, Sushchanskiĭ)

Theorem (D'Angeli, Matucci, Perego, R.)

Thomson $\mathbb{F}$ *is* **CF-TR**.

Proposition (D'Angeli, Matucci, Perego, R.)

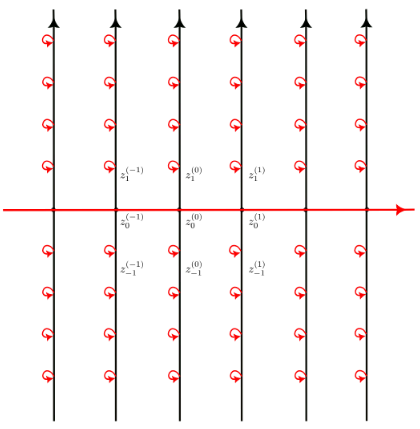
A group G is virtually a finitely generated subgroup of the direct product of free groups if and only if G is the transition group of a collection of quasi-transitive context-free inverse graphs. In particular, if Brough's conjecture holds, then **poly-CF** *would be included in* **CF-TR**.

Transition groups: some structural properties (III)

It would not be equal:

Theorem (D'Angeli, Matucci, Perego, R.)

The following transition group contains $\mathbb{Z}^\infty$, thus in particular it is not **poly-CF**.



Transition groups: local perturbation (I)

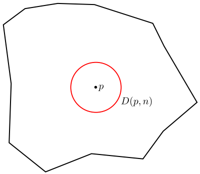
What happens if one “perturbs” the graph?

Definition (Boundary group)

Let $\Lambda_n = \Gamma \setminus D(p,n)$ be the graph in which we cut a disk of radius n.
Define:

$$\mathcal{O}_n = \{u \in A^* : \text{ if } x \xrightarrow{u} x' \text{ is a walk in } \Lambda_n \text{ then } x = x'\}$$

The projection $\pi(\mathcal{O}_n) = \mathcal{B}_n$ into the transition group $G(\Gamma)$ is called the boundary subgroup at level n.



Proposition (D'Angeli, Matucci, Perego, R.)

For all $n \geq 1$, the boundary group $\mathcal{B}_n$ at level n is a torsion normal subgroup of G where each element g has order $o(g) \leq O(|g|)$.

Transition groups: local perturbation (II)

Definition

A graph Γ is locally-quasi-transitive if $\Gamma \setminus D(p,n) = \Theta \setminus D(p',n)$ and Θ is quasi-transitive where each orbit is infinite.

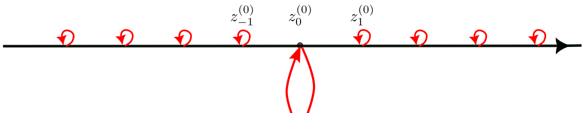
Theorem (D'Angeli, Matucci, Perego, R.)

Let Γ be a context-free inverse graph that is also locally-quasi-transitive. Then, there is a short exact sequence

$$1 \to \mathcal{B}_n \to G(\Gamma) \to H \to 1$$

where H is a group that is virtually a finitely generated subgroup of the direct product of free groups and $\mathcal{B}_n$ is a torsion normal subgroup. In particular we have $H \simeq G(\Gamma)/_{\mathscr{B}_n}$.

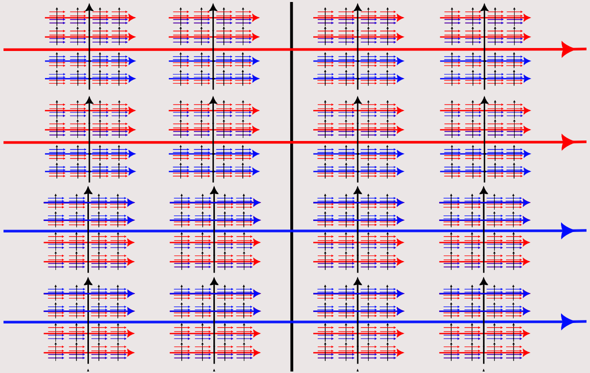
For instance, the boundary subgroup $\mathcal{B}_1$ of the following graph



Thomson $\mathbb{V}$ and **CF-TR**

Conjecture

The transition group of the following context-free graph is not in $\mathbb{V}$. Therefore, Lehner's conjecture is false.



Monoids perspective

Definition

One may define the transition monoid of a context-free inverse graph (in general not complete). It is clearly an inverse monoid.

Some open thoughts

- Is there a notion of **co-CF** (inverse) monoid?
- Is the class of **CF-TR** included in **co-CF** ?
- What is the relationship between the transition monoid of a Schützenberger graph and its original monoid? (it seems a quotient)
- What happens if the Schützenberger graph is context-free?
- What happens if the Schützenberger graph $\Gamma(e)$ is quasi-transitive (i.e., the maximal subgroup $H_e \simeq Aut(\Gamma(e))$ acts transitively on $\Gamma(e)$)?
- How is the local perturbation of the graph reflected algebraically to the transition monoid?


Thank you!