Transformation representations of diagram monoids

James East
Western Sydney University

The team

Reinis Cirpons James East James Mitchell

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- RC+JE+JM (2024): Yes :-)
  - Transformation representations of diagram monoids
    - `arXiv:2411.14693`

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  - Enumeration by size: almost all semigroups are boring :-(
  - Enumeration by degree: almost all semigroups are interesting :-)
- **Many authors** have calculated $\deg(S)$ for various (semi)groups S.
  - Babai, Cain, Cameron, Easdown, Elias, FitzGerald, Hendriksen, Holt, Johnson, Kovács, Malheiro, Margolis, Paulista, Pebody, Praeger, Quinn-Gregson, Saunders, Schein, Steinberg, Wright...

...and us!

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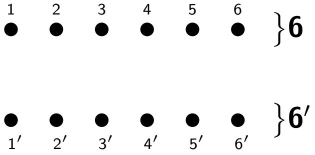

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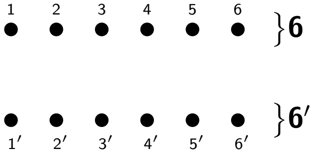

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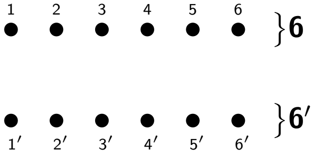

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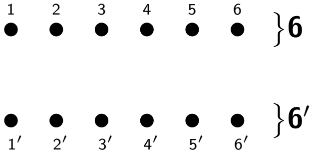

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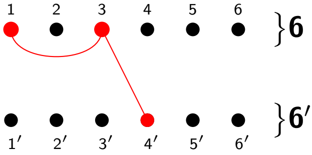

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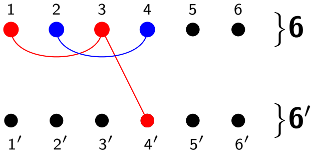

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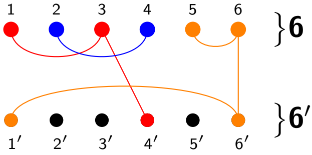

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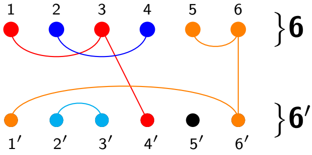

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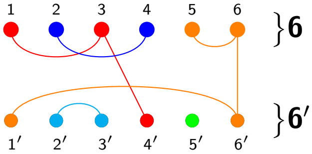

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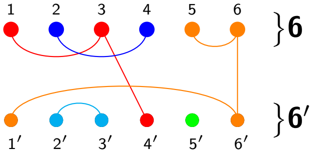

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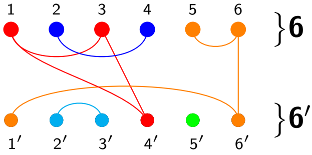

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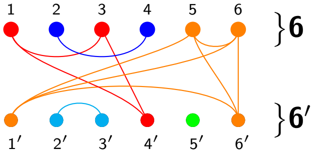
- Eg: $a = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 6'\}, \{2', 3'\}, \{5'\}\Big\} \in \mathcal{P}_6$



Partition monoids — product in $\mathcal{P}_n$

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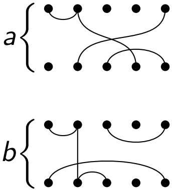

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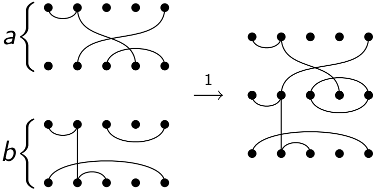

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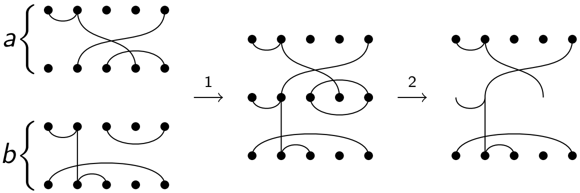

Partition monoids — product in $\mathcal{P}_n$

To calculate the product of $a, b \in \mathcal{P}_n$:

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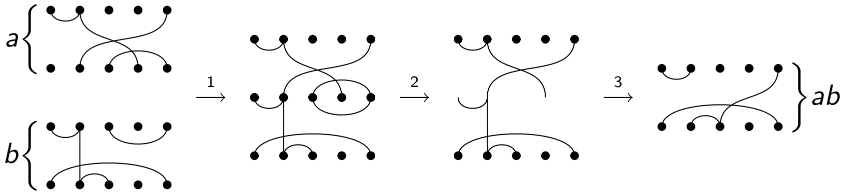
(3) tidy up.



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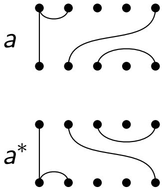

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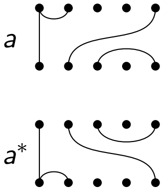

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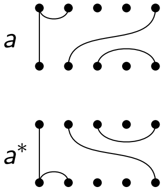

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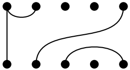
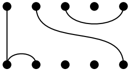
a



a^*



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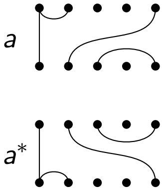

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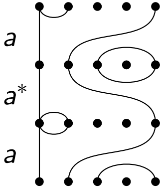

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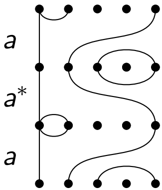

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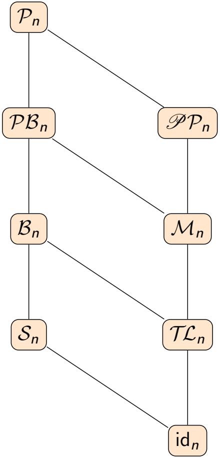
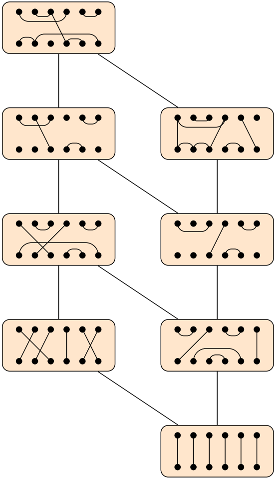
Diagram monoids — submonoids of $\mathcal{P}_n$





- Brauer, Temperley–Lieb, Motzkin, and more......

Diagram monoids — transformation degree

Today's question

What is $\deg(\mathcal{P}_n)$?

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Short answer (add 1 for $\deg(M)$):

Monoid M	Validity	Minimum partial transformation degree $\deg'(M)$
$\mathcal{P}_n$	$n \geq 2$	$\frac{B(n+2)-B(n+1)+B(n)}{2}$
$\mathcal{PB}_n$	$n \geq 2$	$\frac{I(n+2)}{2}$
$\mathcal{B}_n$	$n \geq 3$ odd	$\frac{n+1}{2} \cdot n!!$
	$n \geq 4$ even	$\frac{(n+4)(n+2)}{8} \cdot (n-1)!!$
$\mathscr{P}\mathcal{P}_n$	$n \geq 2$	$C(n+2) - 2C(n+1) + C(n)$
$\mathcal{M}_n$	$n \geq 2$	$M(n+2) - M(n+1)$
$\mathcal{TL}_n$	$n = 2k-1 \geq 3$	$C(k+1) - C(k)$
	$n = 2k \geq 4$	$C(k+2) - 2C(k+1) + C(k)$

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	$n \geq 4$ even	$p_0 + 2p_2 + 3p_4$
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$\mathcal{TL}_n$	$n \geq 3$ odd	$p_1 + p_3$
	$n \geq 4$ even	$p_0 + p_2 + p_4$

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n	0	1	2	3	4	5	6	7	8	9	10	OEIS
$\deg'(\mathcal{P}_n)$	1	1	6	21	83	363	1733	8942	49 484	291 871	1825 501	A087649
$\deg'(\mathcal{PB}_n)$	1	1	5	13	38	116	382	1310	4748	17 848	70 076	A001475
$\deg'(\mathcal{B}_n)$	1		2		18		150		1575		19 845	$\frac{1}{3}\times$A001194
		1		6		45		420		4725		A001879
$\deg'(\mathscr{P}\mathcal{P}_n)$	1	1	6	19	62	207	704	2431	8502	30 056	107 236	A026012
$\deg'(\mathcal{M}_n)$	1	1	5	12	30	76	196	512	1353	3610	9713	A002026
$\deg'(\mathcal{TL}_n)$	1		1		6		19		62		207	A026012
		1		3		9		28		90		A000245

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- Key tools:
  - actions, (one- and two-sided) congruences, projections.

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    - Faithful: different elements of S act differently.

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- If S is a monoid, then this action is monogenic: $[x] = [1]^x$.
  - Conversely, any monogenic monoid action is a right cong. action.
- Key fact: The action of a monoid S on S/σ is faithful

  iff σ contains no non-trivial two-sided congruence.

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- The two-sided congruences of $\mathcal{P}_n$ are known.

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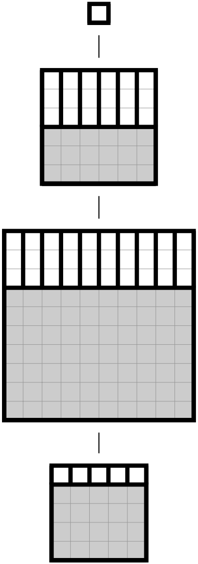

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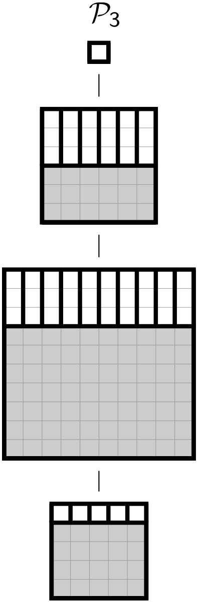

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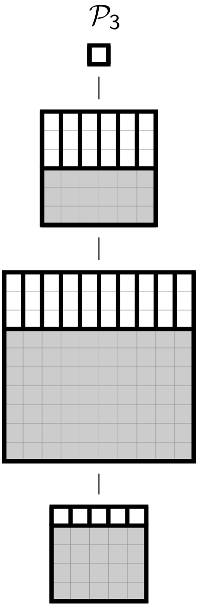

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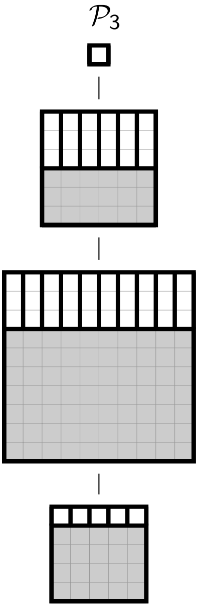

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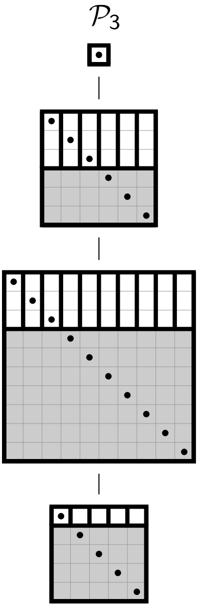

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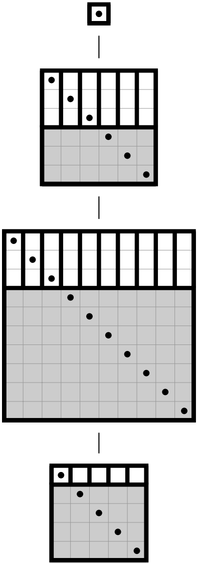

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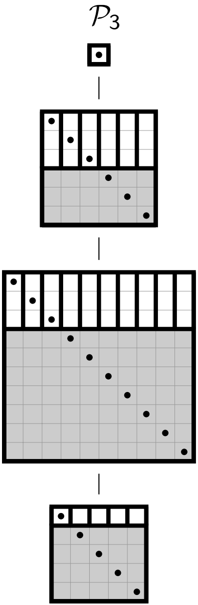

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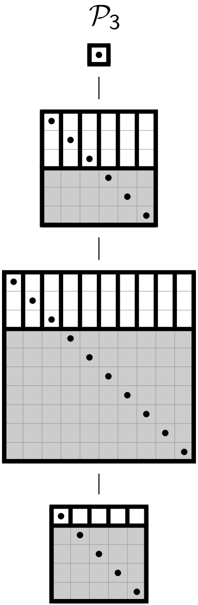

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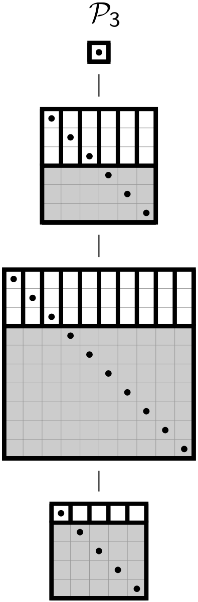

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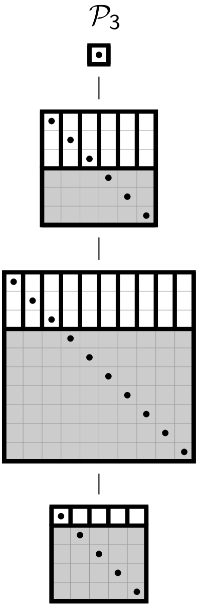

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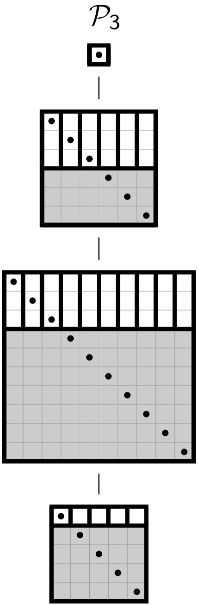

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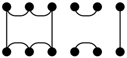
- Sample projection $p \in \mathcal{P}_n$ .... p  .

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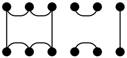
- Sample projection $p \in \mathcal{P}_n$ .... p  .
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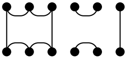
- Sample projection $p \in \mathcal{P}_n$ .... p  .
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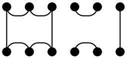
- Sample projection $p \in \mathcal{P}_n$ .... p  .
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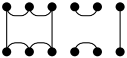
- Sample projection $p \in \mathcal{P}_n$ .... p  .
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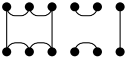
- Sample projection $p \in \mathcal{P}_n$ .... p  .
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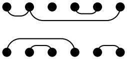
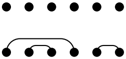
  - e.g. $a =$  and $b =$  act the same.

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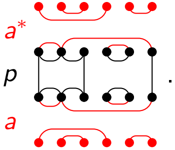

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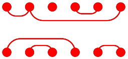
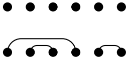
  - e.g. $a =$  and $b =$  act the same.

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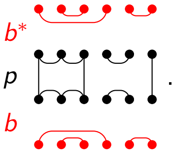

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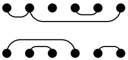
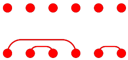
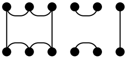
  - e.g. $a =$  and $b =$  act the same.

Tool 3: Partial actions on projections

- Let S be a regular $*$-semigroup:
  - $a = aa^*a$,
  - $a^{**} = a$,
  - $(ab)^* = b^*a^*$.
- Let $P = P(S) = \{p \in S : p^2 = p = p^*\}$.
  - Elements of P are called projections.
- Sample projection $p \in \mathcal{P}_n$ .... p  .
- Fact: $P = \{a^*a : a \in S\}$.
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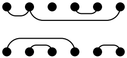
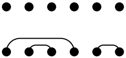
  - e.g. $a =$  and $b =$  act the same.
- Solution: partialise the action.

Tool 3: Partial actions on projections

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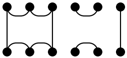
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .

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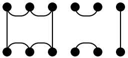
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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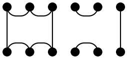
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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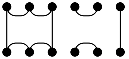
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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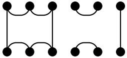
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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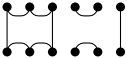
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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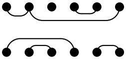
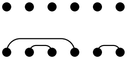
- E.g. again consider $a =$  and $b =$  .

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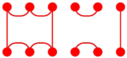
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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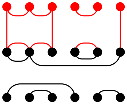
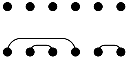
- E.g. again consider $a =$  and $b =$  .
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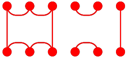
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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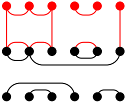
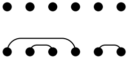
- E.g. again consider $a =$  and $b =$  .
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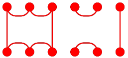
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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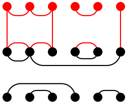
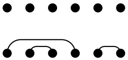
- E.g. again consider $a =$  and $b =$  .
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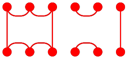
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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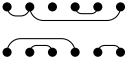
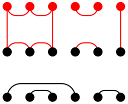
- E.g. again consider $a =$  and $b =$  .
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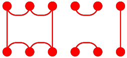
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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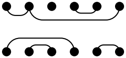
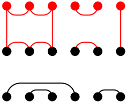
- E.g. again consider $a =$  and $b =$  .
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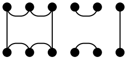
- E.g. $\ker(p) = (1,2,3 \mid 4,5 \mid 6)$ for $p =$  .
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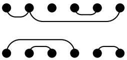
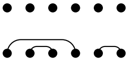
- E.g. again consider $a =$  and $b =$  .
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    - So a and b act differently (on p).

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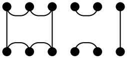
- E.g. $\text{rank}(p) = 2$ for $p =$  .

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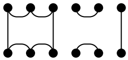
- E.g. $\text{rank}(p) = 2$ for $p =$  .
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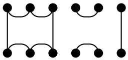
- E.g. $\text{rank}(p) = 2$ for $p =$  .
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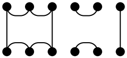
- E.g. rank$(p) = 2$ for $p =$  .
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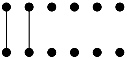
- This action is monogenic, generated by $t =$  .

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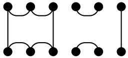
- E.g. $\text{rank}(p) = 2$ for $p =$  .
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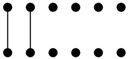
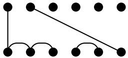
- This action is monogenic, generated by $t =$  .
  - E.g. $p = t^{\widehat{p}}$ where $\widehat{p} =$  .

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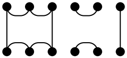
- E.g. $\text{rank}(p) = 2$ for $p =$  .
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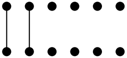
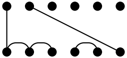
- This action is monogenic, generated by $t =$  .
  - E.g. $p = t^{\widehat{p}}$ where $\widehat{p} =$  .
- The action therefore corresponds to a right congruence

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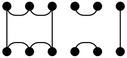
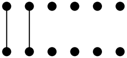
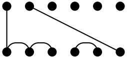
- E.g. $\text{rank}(p) = 2$ for $p =$  .
- We have $\text{rank}(a^* p a) \leq \text{rank}(p)$.
- $Q = \{p \in P : \text{rank}(p) \leq 2\}$ is closed under the partial action.
- This action is monogenic, generated by $t =$  .
  - E.g. $p = t^{\widehat{p}}$ where $\widehat{p} =$  .
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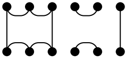
- E.g. $\text{rank}(p) = 2$ for $p =$  .
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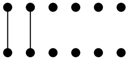
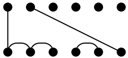
- This action is monogenic, generated by $t =$  .
  - E.g. $p = t^{\widehat{p}}$ where $\widehat{p} =$  .
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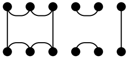
- E.g. $\mathrm{rank}(p) = 2$ for $p =$  .
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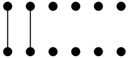
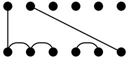
- This action is monogenic, generated by $t =$  .
  - E.g. $p = t^{\widehat{p}}$ where $\widehat{p} =$  .
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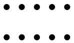
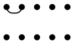
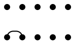
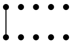
  where $z =$ , $a =$ , $b =$ , $c =$ .

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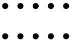
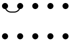
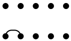
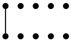
  where $z =$ , $a =$ , $b =$ , $c =$ .
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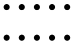
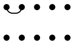
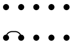
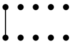
where $z =$  , $a =$  , $b =$  , $c =$  .

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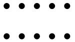
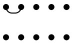
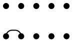
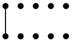
  where $z =$ , $a =$ , $b =$ , $c =$ .
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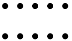
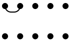
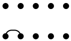
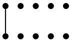
  where $z =$ , $a =$ , $b =$ , $c =$ .
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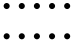
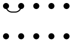
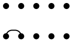
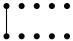
  where $z =$ , $a =$ , $b =$ , $c =$ .

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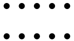
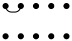
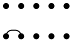
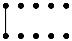
where $z =$ , $a =$ , $b =$ , $c =$ .

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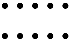
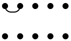
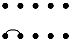
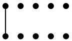
  where $z =$ , $a =$ , $b =$ , $c =$ .

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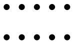
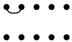
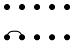
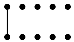
  where $z =$ , $a =$ , $b =$ , $c =$ .
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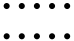
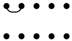
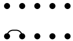
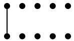
  where $z =$ , $a =$ , $b =$ , $c =$ .
- The action must 'separate' each of the above pairs.
  - So $x_2^z \neq x_2^a$, $x_0^z \neq x_0^b$, $x_1^z \neq x_1^c$ for some $x_0, x_1, x_2 \in X$.
- Let $Y_i = x_i^{\widehat{P_i}} = \{x_i^{\widehat{p}} : p \in P_i\}$. Here $P_i = \{p \in P : \text{rank}(p) = i\}$.
  - Then $|Y_i| = |P_i|$ for all i...... and $Y_0 \sqcup Y_1 \sqcup Y_2 \subsetneq X$.
- So $|X| \geq |P_0| + |P_1| + |P_2| + 1 = 1 + |Q|$.

Degree of $\mathcal{P}_n$

Theorem

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- $\deg(\mathcal{P}_n) = \deg_{\mathrm{rc}}(\mathcal{P}_n) = 1 + |Q|$

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- Things are very different for $\mathcal{B}_n$.

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  - The size of Q involves different combinatorial parameters.
- The proof can be adapted for $\mathcal{TL}_n$.
- Things are very different for $\mathcal{B}_n$.
  - e.g. $\deg(\mathcal{B}_{2k}) < \deg_{\mathrm{rc}}(\mathcal{B}_{2k})$.

Degree of diagram monoids

Monoid M	Validity	Minimum partial transformation degree $\deg'(M)$
$\mathcal{P}_n$	$n \geq 2$	$p_0 + p_1 + p_2$
$\mathcal{PB}_n$	$n \geq 2$	$p_0 + p_1 + p_2$
$\mathcal{B}_n$	$n \geq 3$ odd	$p_1 + 3p_3$
	$n \geq 4$ even	$p_0 + 2p_2 + 3p_4$
$\mathscr{P}\mathcal{P}_n$	$n \geq 2$	$p_0 + p_1 + p_2$
$\mathcal{M}_n$	$n \geq 2$	$p_0 + p_1 + p_2$
$\mathcal{TL}_n$	$n \geq 3$ odd	$p_1 + p_3$
	$n \geq 4$ even	$p_0 + p_2 + p_4$

Degree of diagram monoids

Monoid M	Validity	Minimum partial transformation degree $\deg'(M)$
$\mathcal{P}_n$	$n \geq 2$	$\frac{B(n+2)-B(n+1)+B(n)}{2}$
$\mathcal{PB}_n$	$n \geq 2$	$\frac{I(n+2)}{2}$
$\mathcal{B}_n$	$n \geq 3$ odd	$\frac{n+1}{2} \cdot n!!$
	$n \geq 4$ even	$\frac{(n+4)(n+2)}{8} \cdot (n-1)!!$
$\mathscr{P}\mathcal{P}_n$	$n \geq 2$	$C(n+2) - 2C(n+1) + C(n)$
$\mathcal{M}_n$	$n \geq 2$	$M(n+2) - M(n+1)$
$\mathcal{TL}_n$	$n = 2k-1 \geq 3$	$C(k+1) - C(k)$
	$n = 2k \geq 4$	$C(k+2) - 2C(k+1) + C(k)$

Thanks for listening :-)

Reinis Cirpons James East James Mitchell

- Transformation representations of diagram monoids
  - `arXiv:2411.14693`