Finitary conditions for graph products

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Finitary conditions

Finitary conditions

A finitary condition for a class of algebras $\mathcal{A}$ is a property, defined for algebras in $\mathcal{A}$, which is satisfied by all finite algebras in $\mathcal{A}$.

Examples

- **Algebras** Being finitely generated
- **Groups** Every element has finite order
- **Groups** Being finitely generated and every element has finite order
- **Semigroups/Monoids** $\mathcal{D} = \mathcal{J}$
- **Semigroups/Monoids/Rings** There are no infinite strictly ascending chains of right ideals

$$I_1 \subseteq I_2 \subseteq \cdots \ldots$$

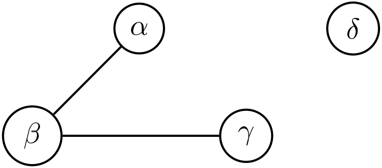
equivalently, every right ideal is finitely generated.

What are graph products all about?

- The construction of a **graph product** involves controlled commutation.
- It allows us to **glue together** monoids (and semigroups, groups, inverse semigroups, ...) to produce new ones, in such a way that certain elements commute.
- Graph products provide a common framework for the *algebraic notions* of direct and free products.
- The notion of a monoid graph product in the case the constituent monoids are **free** appears in *computer science*. They are called **trace monoids, free partially commutative monoids, (heaps)** and provide a model of parallel computation.
- Graph products of free *groups* are **RAAGs, free partially commutative groups, semifree groups,...**

Graph products of monoids

Let $\Gamma = (V, E)$ be a simple undirected graph.



Let $\mathcal{M} = \{M_\alpha : \alpha \in V\}$ be a set of mutually disjoint monoids, called **vertex monoids**; we write 1_α for the identity of M_α.

The graph product

of $\mathcal{M}$ with respect to Γ is the 'freest' monoid generated by submonoids M_α's, such that elements of 'adjacent' vertex monoids commute.

We approach this via a presentation.

Definition of graph products: da Costa (2002)

For a set X, we let $X^* = \{x_1 \circ \cdots \circ x_n : n \in \mathbb{N}^0, x_i \in X\}$.

The graph product $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathcal{M})$ of $\mathcal{M}$ with respect to Γ

$$\mathscr{GP} = \langle X \mid R \rangle$$

where $X = \bigcup_{\alpha \in V} M_\alpha$ and with defining relations $R = R_{\mathrm{v}} \cup R_{\mathrm{e}} \cup R_{\mathrm{id}}$:

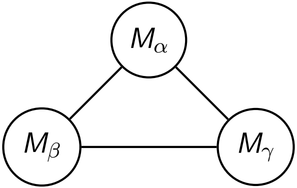
(R_{v}) $x \circ y = xy$ $(x, y \in M_\alpha, \alpha \in V)$;

(R_{e}) $x \circ y = y \circ x$ $(x \in M_\alpha, y \in M_\beta, (\alpha, \beta) \in E)$;

(R_{id}) $1_\alpha = \epsilon$, $\alpha \in V$.

Two elements $u, v \in X^*$ are equal in $\mathscr{GP}$, written $u \equiv v$, if and only if one can get from one to the other by substituting a LHS/RHS of a relation by the RHS/LHS.

What does this mean? - Complete graph, i.e. $E = V \times V$

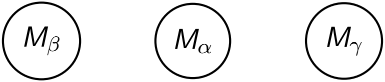


In $\mathscr{G}\mathscr{P}$ how do we handle (with a natural convention for labelling, i.e. $s_\alpha \in M_\alpha$, etc.) the word $s_\alpha \circ s_\beta \circ s'_\beta \circ s_\gamma \circ s''_\beta$? We have

$$\begin{aligned} s_\alpha \circ s_\beta \circ s'_\beta \circ s_\gamma \circ s''_\beta &\equiv s_\alpha \circ (s_\beta \circ s'_\beta) \circ s_\gamma \circ s''_\beta \\ &\equiv s_\alpha \circ s_\beta s'_\beta \circ s_\gamma \circ s''_\beta \\ &\equiv s_\alpha \circ s_\beta s'_\beta \circ (s_\gamma \circ s''_\beta) \\ &\equiv s_\alpha \circ s_\beta s'_\beta \circ (s''_\beta \circ s_\gamma) \\ &\equiv s_\alpha \circ s_\beta s'_\beta s''_\beta \circ s_\gamma . \end{aligned}$$

We have $\mathscr{G}\mathscr{P} \cong M_\alpha \times M_\beta \times M_\gamma$.

What does this mean? - Null graph, i.e. $E = \emptyset$



In $\mathscr{G}\mathscr{P}$ how do we handle (with a natural convention for labelling, i.e. $s_\alpha \in M_\alpha$, etc.) the word $s_\alpha \circ s_\gamma \circ s_\beta \circ s'_\beta \circ s'_\gamma \circ s''_\beta$? We have

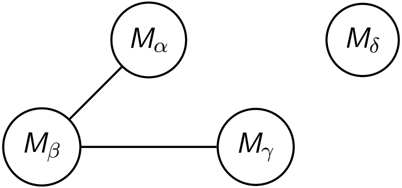
$$\begin{aligned} s_\alpha \circ s_\gamma \circ s_\beta \circ s'_\beta \circ s_\gamma \circ s''_\beta &\equiv s_\alpha \circ s_\gamma \circ (s_\beta \circ s'_\beta) \circ s'_\gamma \circ s''_\beta \\ &\equiv s_\alpha \circ s_\gamma \circ s_\beta s'_\beta \circ s'_\gamma \circ s''_\beta \end{aligned}$$

if no elements are identities, in particular, $s_\beta s'_\beta \neq I_\beta$, then we can do no more. If $s_\beta s'_\beta = I_\beta$

$$\begin{aligned} s_\alpha \circ s_\gamma \circ s_\beta s'_\beta \circ s'_\gamma \circ s''_\beta &\equiv s_\alpha \circ s_\gamma \circ s'_\gamma \circ s''_\beta \\ &\equiv s_\alpha \circ s_\gamma s'_\gamma \circ s''_\beta \cdots \end{aligned}$$

For a null graph we have $\mathscr{G}\mathscr{P} \cong M_\alpha * M_\beta * M_\gamma$ where $*$ is free product.

What does this mean? ....in general



In $\mathscr{G}\mathscr{P}$ how do we handle (with a natural convention for labelling, i.e. $s_\alpha \in S_\alpha$, etc.) the word $s_\alpha \circ s_\beta \circ s_\delta \circ s'_\delta \circ s'_\beta \circ s_\gamma \circ s''_\beta$? We have

$$\begin{aligned}
s_\alpha \circ s_\beta \circ s_\delta \circ s'_\delta \circ s'_\beta \circ s_\gamma \circ s''_\beta &\equiv s_\alpha \circ s_\beta \circ s_\delta s'_\delta \circ s'_\beta \circ s_\gamma \circ s''_\beta \\
&\equiv s_\alpha \circ s_\beta \circ s_\delta s'_\delta \circ (s'_\beta \circ s_\gamma) \circ s''_\beta \\
&\equiv s_\alpha \circ s_\beta \circ s_\delta s'_\delta \circ (s_\gamma \circ s'_\beta) \circ s''_\beta \\
&\equiv s_\alpha \circ s_\beta \circ s_\delta s'_\delta \circ s_\gamma \circ (s'_\beta \circ s''_\beta) \\
&\equiv s_\alpha \circ s_\beta \circ s_\delta s'_\delta \circ s_\gamma \circ (s'_\beta s''_\beta) \\
\text{(and if } s_\delta s'_\delta = 1_\delta) &\equiv s_\alpha \circ s_\beta \circ s_\gamma \circ s'_\beta s''_\beta \ldots.
\end{aligned}$$

Some technicalities

- Two words $x_1 \circ \cdots \circ x_m$ and $y_1 \circ \cdots \circ y_m$ are **shuffle equivalent** if you can reach one from the other only using relations (R_e).
- Words of least length in a congruence class are **reduced** and are all shuffle equivalent.
- Any word w has **right Foata normal form**: a reduced $w' \equiv w$ with

$$w' = w_1 \circ \cdots \circ w_n$$

  each w_i is the maximum block with complete support, reading from right to left; dually, there is a in **left Foata normal form**. Left/right Foata normal forms are unique up to inter-block shuffling.
- Starting with reduced words w, a, understanding the process of reducing $w \circ a$ is not straightforward.

What kind of questions should we ask about graph products?

What properties pass up and down to the graph product?

Given $\mathscr{G}\mathscr{P}(\Gamma, \mathcal{M})$ has property X, does each vertex monoid have property X?
If each vertex monoid has property X, does $\mathscr{G}\mathscr{P}(\Gamma, \mathcal{M})$ have property X?

- **Algorithmic properties** do graph products have, depending on the properties of the ingredients?
- **Algebraic properties** Regularity, cancellativity, abundancy, etc.
- **Finitary conditions** Residual finiteness, chain conditions, etc.

What kind of questions should we ask about graph products: finitary conditions
Residual finiteness

Residual finiteness

A monoid M is residually finite if for any distinct $a, b \in M$ there is a finite monoid F and a morphism $\theta : M \to F$ such that $a\theta \neq b\theta$.

Theorem: Cho, G, Ruškuc, Yang (2024)

A graph product is residually finite if and only if each vertex monoid is residually finite.

Finitary conditions for graph products: noetherianity and coherency

A monoid M is

- **right noetherian** if every right congruence is finitely generated
- **right coherent** if every finitely generated subact of every finitely presented M-act is finitely presented.

These notions follow those for rings, where we consider right congruences in place of right ideals.

Finitary conditions for graph products: noetherianity and coherency

The good news

If $\mathcal{GP}(\Gamma, \mathcal{M})$ is right noetherian/coherent, then so is every vertex monoid.

This follows from:

A submonoid M of N is a **retract** of N if there is an onto morphism $\theta : N \to M$ such that $\theta^2 = \theta$ (equivalently, $\theta|_M : M \to M$ is the identity on M).

Easy fact: every vertex monoid is a retract of $\mathcal{GP}(\Gamma, \mathcal{M})$.

Fact

The class of right noetherian monoids **(Miller and Ruškuc (2019))** and right coherent monoids **(G. and Hartmann (2017))** are each closed under retract.

Finitary conditions for graph products: noetherianity and coherency

The bad news

- Let F_3 be the free monoid on 3 generators. Then F_3 is right coherent, but $F_3 \times F_3$ is not (**G, Hartmann, Ruškuc (2017)**).
- It is not known that $M \times N$ is right noetherian for arbitrary right noetherian M, N.

Finitary conditions for graph products: the conditions for today

A monoid M

- is **weakly right noetherian** if there are no infinite strictly ascending chains of right ideals; equivalently, every right ideal is finitely generated as a right ideal;
- is **right ideal Howson** if the intersection of finitely generated right ideals is finitely generated;
- satisfies **ACCPR** if it has no infinite strictly ascending chains of principal right ideals;
- is **finitely right equated (FRE)** if $\mathbf{r}(a) := \{(u, v) : au = av\}$ is finitely generated as a right congruence, for all $a \in M$;
- is **weakly right coherent** if every finitely generated right ideal has a finite presentation.

Right ideal Howson monoids are also called **finitely right aligned**.

First thoughts on these conditions

We have the following implications between our conditions:

Right noetherian $\Rightarrow$ weakly right noetherian $\Rightarrow$ ACCPR & right ideal Howson

Normak (1977)

Right noetherian $\Rightarrow$ Right coherent

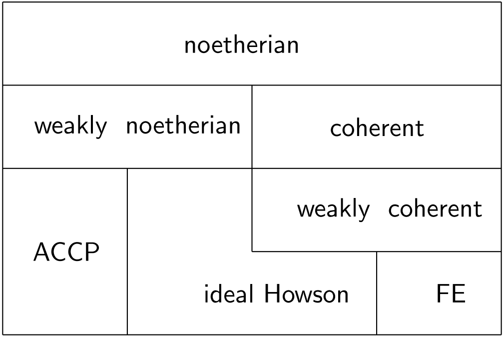
G. (1992)

Right coherent $\Rightarrow$ weakly right coherent $\Leftrightarrow$ finitely right equated and right ideal Howson

The easy good news

If $\mathcal{GP}(\Gamma, \mathcal{M})$ satisfies any of the conditions above, then so does every vertex monoid.

Connections between all the conditions



weakly coherent
= ideal Howson + FE

Weak right noetherianity - what we knew

Miller (2021)

Let M, N be monoids. Then

- the free product of M and N is weakly right noetherian if and only if either M and N are groups, or $|M| = |N| = 2$.
- The direct product of M and N is weakly right noetherian if and only if both M and N are weakly right noetherian.

Weak right noetherianity

G., Yang (2025)

A graph product $\mathscr{G}\mathscr{P}(\Gamma, \mathcal{M})$ is weakly right noetherian if and only if: all the vertex monoids are weakly right noetherian and

- only finitely many vertex monoids are not groups
- there is at most one pair $(\alpha, \beta) \notin E$ such that M_α, M_β are not both groups and in this case $|M_\alpha| = |M_\beta| = 2$ and $(\alpha, \gamma), (\beta, \gamma) \in E$ for all $\gamma \in V$.

In this case,

$$\mathcal{GP} \cong M \times P \times G \text{ or } P \times G$$

where M is the free product of two two element monoids, P is a finite direct product of non-group right noetherian monoids, and G is a group.

Ascending chain condition on principal right ideals (ACCPR)

Stopar (2012)

A finite direct product of two monoids M and N has ACCPR if and only if so do M and N.

Miller (2023)

A free product of monoids $\{M_i : i \in I\}$ has ACCPR if and only if so does each $M_i, i \in I$.

G., Yang (2025)

A graph product $\mathscr{GP}(\Gamma, \mathcal{M})$ has ACCPR if and only if so does every vertex monoid.

In considering $[w]\mathscr{GP}$, we successively shuffle right invertible elements to the end of w and remove them without affecting $[w]\mathscr{GP}$. So we may suppose w has a r.f.n.f. with no right invertible elements in the final block.

Right ideal Howson

Carson, G. (2021)

A free product or a finite direct product of right ideal Howson monoids is right ideal Howson.

Let $[a], [b] \in \mathscr{GP}$. As earlier, given we are focussing on a right ideal $[a]\mathscr{GP} \cap [b]\mathscr{GP}$, we can assume a, b have a 'nice' right Foata normal form. We want to consider the possibilities for $[a][c] = [b][d]$. We only need to shuffle and glue.

G., Yang (2025)

A graph product of monoids is right ideal Howson if and only if so is every vertex monoid.

Finitely right equated

Recall that M is finitely right equated if for any $a \in M$ we have

$$\mathbf{r}(a) = \{(u, v) : au = av\}$$

is finitely generated as a right congruence.
Thus in considering $au = av$ we are focusing on u and v and not just on the right ideal au generates.

Dasar, G. Miller (2024)

A free product or a finite direct product of monoids that are finitely right equated is finitely right equated.

Finitely right equated

The problem: we need to find a finite set of generators for the right annihilator of $[a] \in \mathscr{G}\mathscr{P}$. In considering $[a][u] = [a][v]$ we need to consider $[u]$ and $[v]$ - we cannot start by manipulating the form of a. The letters can shuffle, glue and delete, and all in different ways.

G., Yang (2025)

A graph product of monoids is finitely right equated if and only if so is every vertex monoid.

Weakly right coherent

Hence we have the corollary we were seeking:

G., Yang (2025)

A graph product of monoids is weakly right coherent if and only if so is every vertex monoid.

Where to from here?

- Finitary conditions associated with descending chains.
- The closure questions for right noetherian monoids; in particular, does there exist right noetherian monoids M, N such that $M \times N$ is not right noetherian?
- The closure questions for right coherent monoids; in particular, given a right coherent monoid, can we determine the class of right noetherian monoids N such that $M \times N$ is right coherent?

Thank you!