Submonoid Membership, finite extensions, and lamplighter groups

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May 2026

based on joint work with Doron Shafrir

Membership problems

Let G be an group (infinite, finitely generated, with decidable Word Problem). Consider the following **membership** problems:

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Definition (Rational Subset Membership)

Input: $g \in G$ and a **rational subset** $S \subseteq G$.
Question: whether $g \in S$?

rational subset = set defined by a rational expression in G
= set recognized by a finite state automaton over G

Relation between the three membership problems

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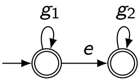

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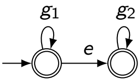

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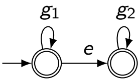

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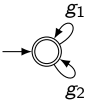

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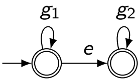

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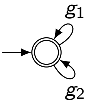

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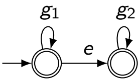

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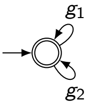

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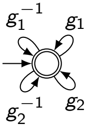

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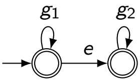

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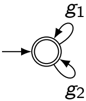

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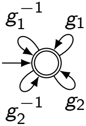

Subgroup Membership $\leq$ Submonoid Membership $\leq$ Rational Subset Mshp.

Classic decidability results

Theorem (Benois 1969, Grunschlag 1999)

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Theorem (Malcev 1958, Romanovskii 1974, Roman'kov 2022)

Subgroup Membership is decidable in f.g. **nilpotent** *and* **metabelian** *groups. But there are nilpotent and metabelian groups where Submonoid Membership is undecidable. (An example is $\mathrm{H}_3(\mathbb{Z})^{5000}$.)*

Hence, Subgroup Mshp. $\lneq$ Submonoid Mshp. $\lneq$ Rational Subset Mshp.

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Problem

What about **Submonoid Membership**?

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Theorem (Shafrir 2024 + D. 2025)

There exists a group G and a finite index subgroup H, such that **Submonoid Membership** *is undecidable in G but decidable in H.*

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Definition (n-dimensional lamplighter group)

The *n-dimensional lamplighter group* $\mathbb{Z}_{/2} \wr \mathbb{Z}^n$ is defined as a matrix group

$$\left\{ \begin{pmatrix} X_1^{z_1} X_2^{z_2} \cdots X_n^{z_n} & f \\ 0 & 1 \end{pmatrix} \middle| \; z_1, \ldots, z_n \in \mathbb{Z}, \underbrace{f \in \mathbb{F}_2[X_1^{\pm}, \ldots, X_n^{\pm}]}_{\text{Laurent polynomial over } \mathbb{F}_2} \right\}.$$

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Each element is a $\mathbb{Z}^2$-grid of lamps, each on ○ or off ●, along with a person 🯅 .

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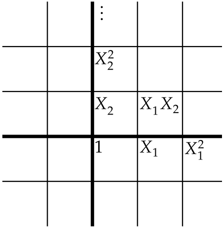

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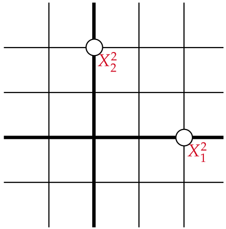

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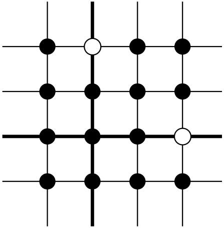

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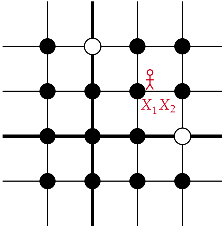

Multiplication in the 2-dimensional lamplighter group

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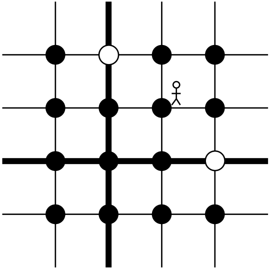

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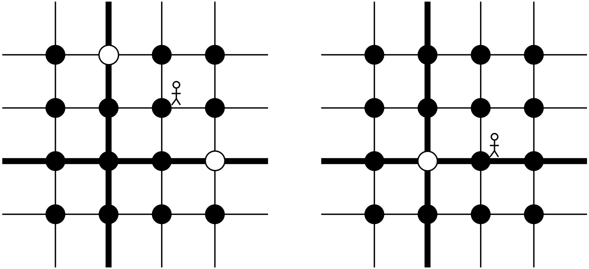
$$\begin{pmatrix} X_1 X_2 & X_1^2 + X_2^2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} X_1 & 1 \\ 1 & 1 \end{pmatrix}$$



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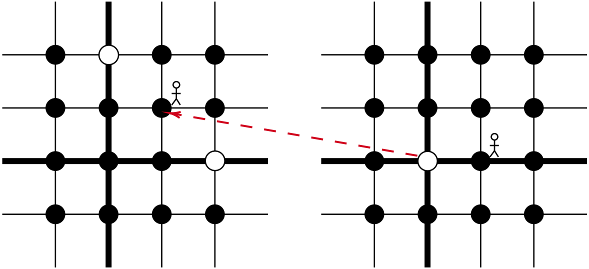

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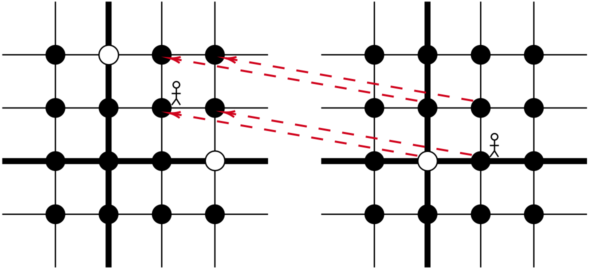

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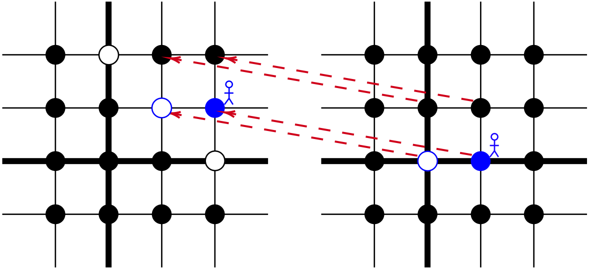

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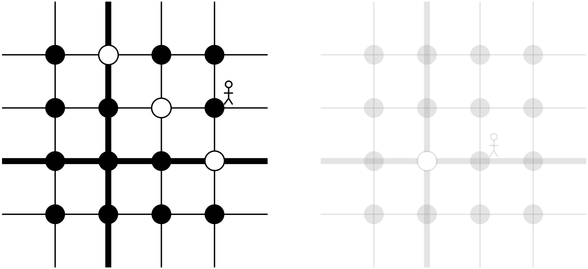
and do pointwise addition on all lamps.

Multiplication in the 2-dimensional lamplighter group

$$= \begin{pmatrix} X_1^2 X_2 & X_1^2 + X_1 X_2 + X_2^2 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} X_1 & 1 \\ 1 & 1 \end{pmatrix}$$



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Submonoid Membership *is decidable in the* 1*-dimensional lamplighter group.*

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Proof is specific to each dimension.

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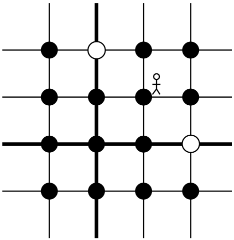
Theorem (D. 2025)

Submonoid Membership *is decidable in the n-dimensional lamplighter group for all n, as well as semidirect products $\mathcal{Y} \rtimes \mathbb{Z}^n$, where $\mathcal{Y}$ is a finitely presented $\mathbb{F}_2[X_1^{\pm}, \dots, X_n^{\pm}]$-module.*

Graphic illustration of semidirect products

$$\left\{ \begin{pmatrix} X_1^{z_1} X_2^{z_2} & f \\ 0 & 1 \end{pmatrix} \;\middle|\; z_1, z_2 \in \mathbb{Z},\;\; \underbrace{f \in \mathbb{F}_2[X_1^{\pm}, X_2^{\pm}]}_{\text{Laurent polynomial over } \mathbb{F}_2} \right\}.$$



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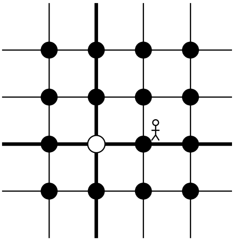

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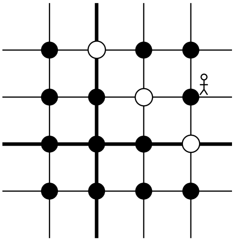
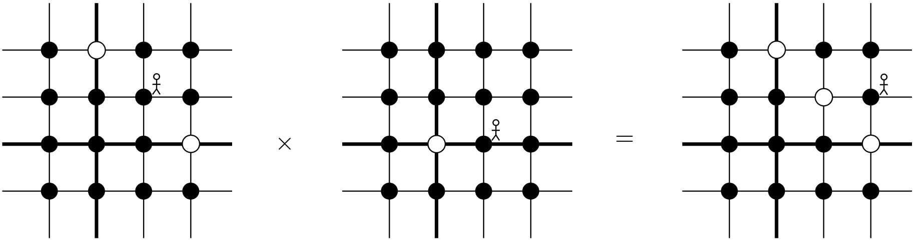


Graphic illustration of semidirect products

$$\left\{ \begin{pmatrix} X_1^{z_1} X_2^{z_2} & f \\ 0 & 1 \end{pmatrix} \,\middle|\, z_1, z_2 \in \mathbb{Z},\ \underbrace{f \in \mathbb{F}_2[X_1^{\pm}, X_2^{\pm}]}_{\text{Laurent polynomial over } \mathbb{F}_2} \right\}.$$



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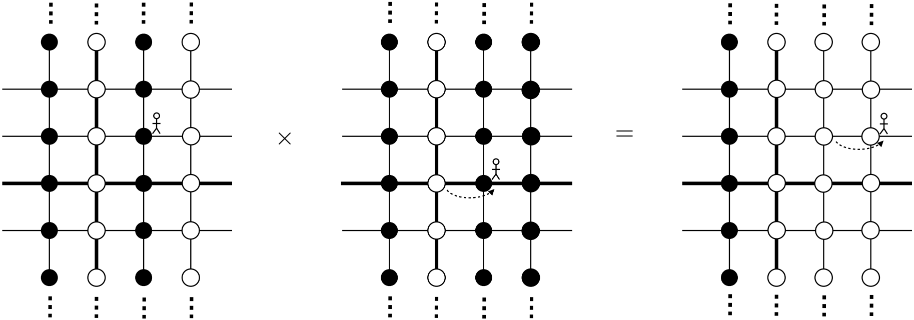

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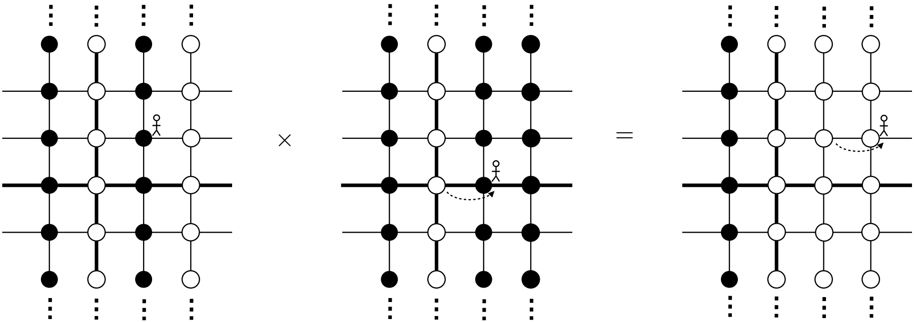

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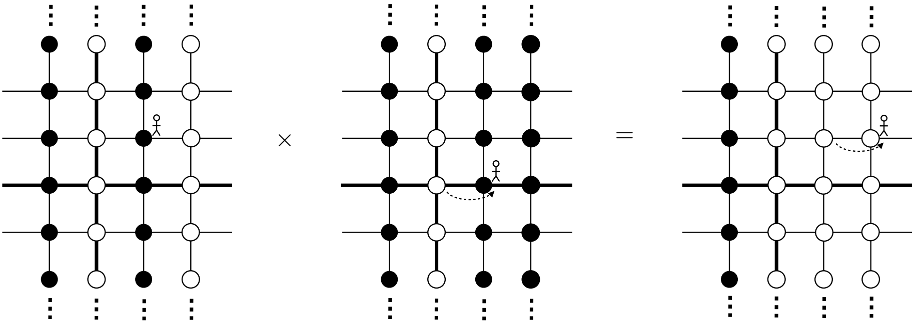

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different "wiring" patterns $\approx$ semidirect products $\mathcal{Y} \rtimes \mathbb{Z}^2$ for different $\mathcal{Y}$

The counterexample

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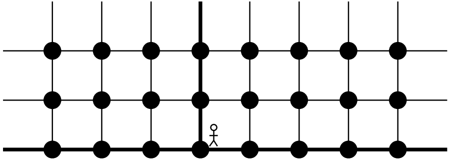

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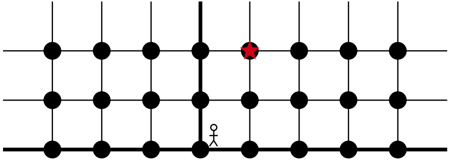

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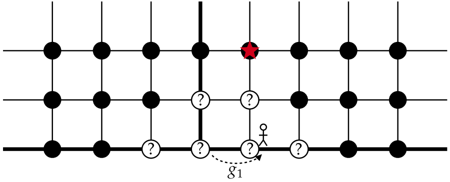

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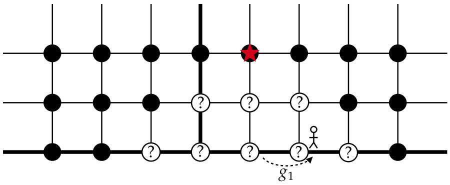

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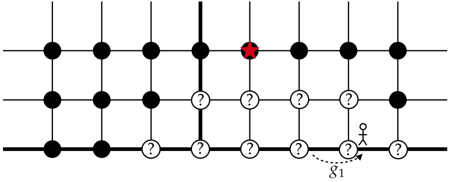

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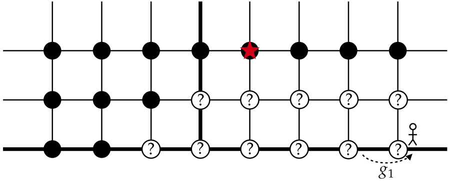

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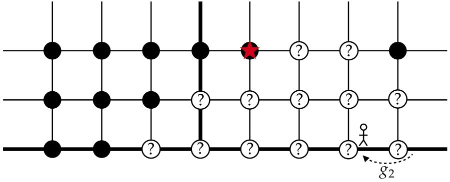

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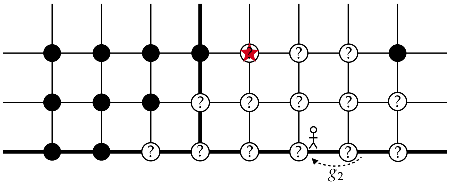

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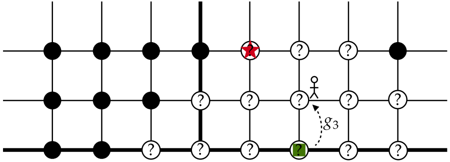

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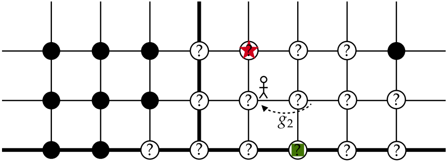

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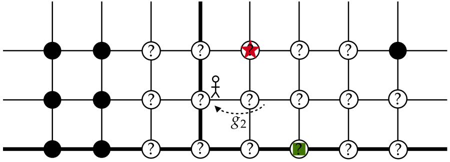

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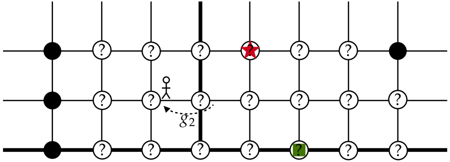

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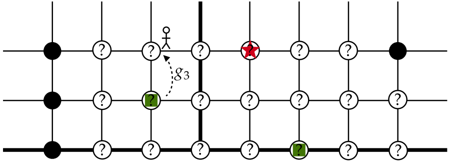

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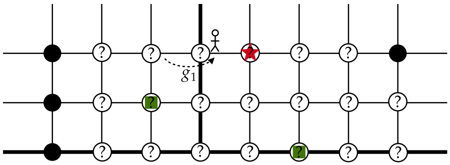

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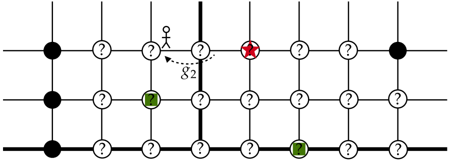

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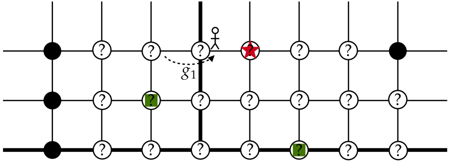

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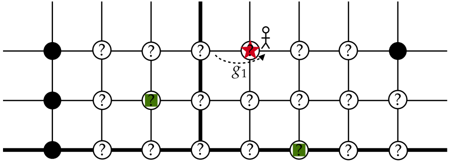

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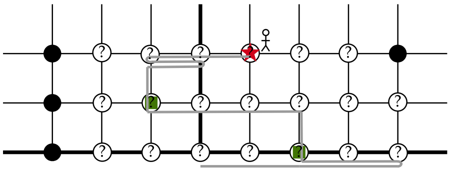

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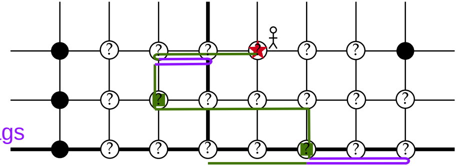

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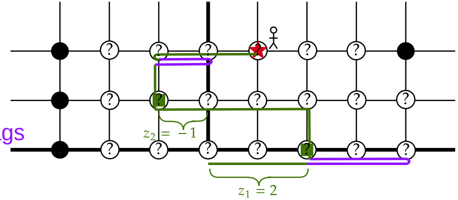

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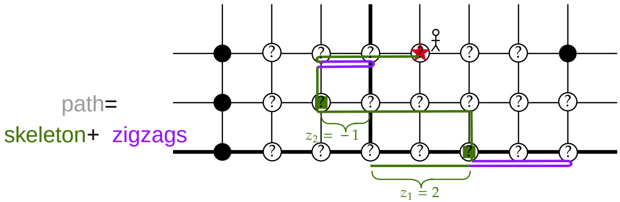

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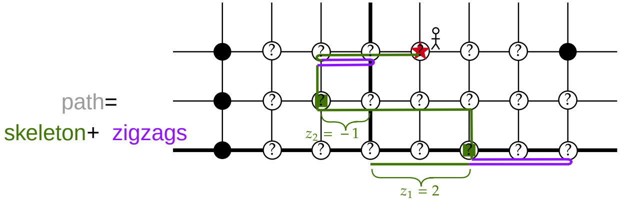

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Submonoid Membership $\rightarrow$ finding $z_1, z_2 \in \mathbb{Z}$, such that $X_1^{z_1} \cdot y_1 + X_1^{z_2} \cdot y_2 = h$ in a quotient module $\mathcal{Y}'$.

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The solution set of an **S-unit equation** *over a finitely presented $\mathbb{F}_2[X_1^{\pm}, \ldots, X_n^{\pm}]$-module is effectively 2-automatic.*

2-automatic: there is an automaton over the alphabet $\{0, 1\}^{kn}$ that recognizes the binary expansion of the solution set.

Example of a solution set

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The solution set is

$$(z_{11}, z_{12}, z_{21}, z_{22}) \in \{(2^k, 0, 0, 2^k) \mid k \in \mathbb{N}\} \cup \{(0, 2^k, 2^k, 0) \mid k \in \mathbb{N}\},$$

which is 2-automatic.

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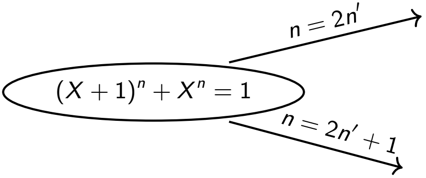

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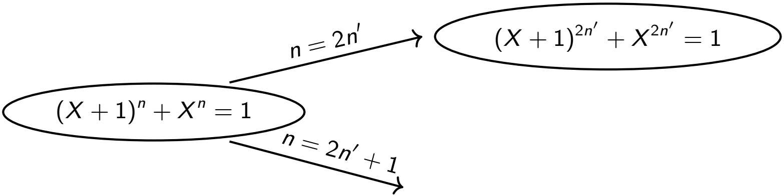

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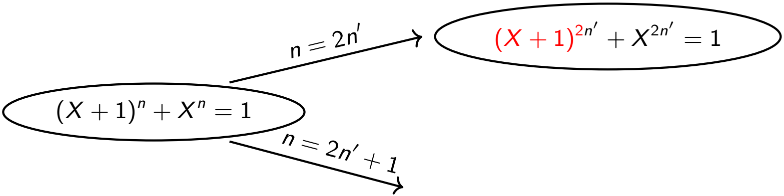

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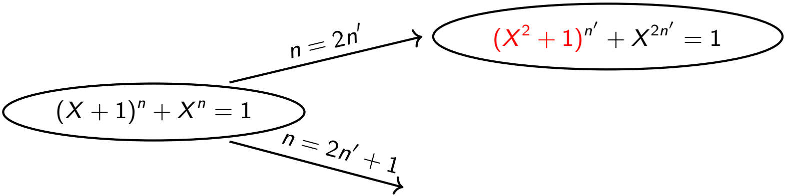

$^*(X+1)^2 = X^2 + 1$: "freshman's dream"

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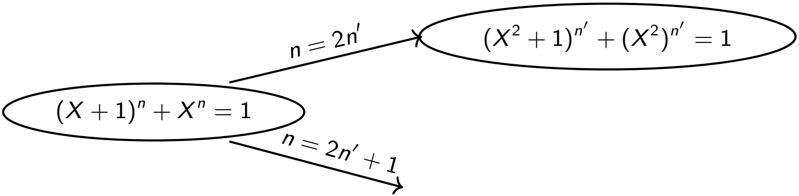

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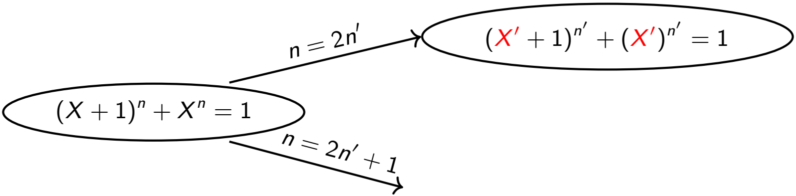

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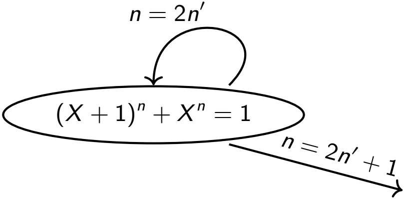

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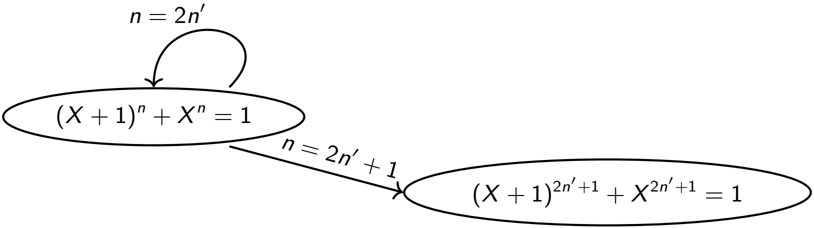

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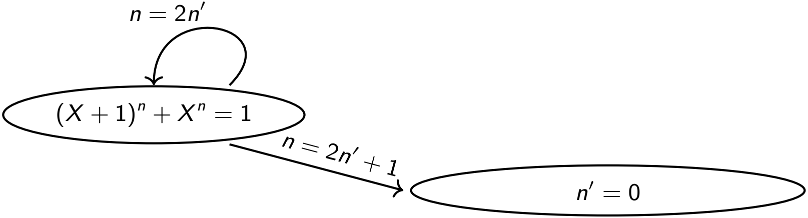

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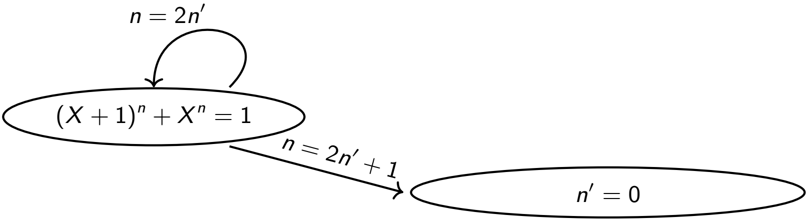

From this automaton, we see directly that n must be of the form 2^k, $k \in \mathbb{N}$.

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As a corollary, we obtained that decidability of Submonoid Membership is not stable under finite extension of groups.

Further work and decidability limits

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