Fracturing infinity: local embeddability

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19.05.26

Fundamentals

Idea

Consider structure (model) S, class $\mathcal{K}$.
For every finite $H \subseteq S$ can we find $F \in \mathcal{K}$ such that H "looks like" a part of F?

Model theory roots

Universal statement: $\forall x_1, \ldots, x_n \; L(x_1, \ldots, x_n)$.
Universal theory $\mathrm{Th}_\forall(\mathcal{K})$: all universal statements which hold in $\mathcal{K}$.
If H "looks" like a part of F and $\forall x_1, \ldots, x_n \; L(x_1, \ldots, x_n) \in \mathrm{Th}_\forall(\mathcal{K})$, then

$$\forall x_1 \in H, \ldots, x_n \in H \; L(x_1, \ldots, x_n)$$

"Converse" is also true.

Literary journey

Model theory

- A. Malcev, *Algebraic systems*, 1973 (with early implicit use in *On isomorphic matrix representations of infinite groups*, 1940)
- O. Belegradek, *Local embeddability*, 2012

Group theory

- T. Evans, *Some connections between residual finiteness, finite embeddability and the word problem*, 1969
- * E. Gordon, A. Vershik, *Groups that are locally embeddable in the class of finite groups*, 1997
- V. Pestov, A. Kwiatkowska, *An introduction to hyperlinear and sofic groups*, 2012
- A. Ould Houcine, F. Point, *Alternatives for pseudofinite groups*, 2013

Acknowledgments and outline

For semigroups see DK, *Semigroups locally embeddable into the class of finite semigroups*, *Restrictions on local embeddability into finite semigroups*, *Effects of semigroup properties on local embeddability*.
Research performed with advice and assistance from Prof Mark Kambites, Dr Marianne Johnson and Prof Victoria Gould.

Plan

- Definition(s) and examples
- Structural properties
- Interactions with other conditions
- New properties
- Open problems

Basic notions

LEF (semi)group S

For every finite subset H of S there exists a finite (semi)group F_H and an injective map $f_H : H \to F_H$, such that for all $x, y \in H$ (with $xy \in H$) we have $(xy)f_H = (xf_H)(yf_H)$. The pair (F_H, f_H) is called *approximating* for H.

Equivalent definitions

Let $\mathcal{F}$ be the class of finite semigroups.

- S is LEF;
- S is a model of $Th_\forall(\mathcal{F})$;
- S is embeddable into a model of $Th(\mathcal{F})$.

Example and non-example

Free semigroup $S_n = Sg\langle x_1, \ldots, x_n \rangle$

Any finite subset H of S_n embeds into the semigroup

$$F_H = S_n / I,$$

where I is the ideal of words longer than elements of H.

Semigroup $T = Sg\langle a, b | a^2 b = a \rangle$

The subset $H = \{a, b, ab, aba\}$ cannot embed into a finite semigroup.
For a potential approximating pair (F, f) of H denote $c = af$ and $d = bf$.
$c^n = c^{n+r} \implies c^{n-1} = c^n d = c^{n+r} d = c^{n+r-1} \implies \ldots \implies c = c^{r+1} \implies cd = c^r \implies cdc = ccd = c$ which contradicts $aba \neq a$.

Further constructions

Embeddable

- Any semigroup with homogenous relations;
- Free groups;
- $Sg\langle a, b, c | ab^n c = b^m \rangle$ for any integer $m, n \geq 1$.

Non-embeddable

- $Sg\langle a, b, c | abc = b, ab^2 c = b^2 \rangle$;
- Bicyclic monoid $B = Mon\langle p, q | pq = 1 \rangle$;
- $T_n = Sg\langle a, b \mid (ba)(ab)^n = (ba)^n \rangle$.

“Universal” in algebra

What statements are in $Th_\forall(\mathcal{F})$?

Generic

$$\forall x, y, z \; (xy)z = x(yz)$$

We cannot express ideas like "the structure is finite" in this language.

Structural

$$\forall a, b \; a^2 b = a \implies aba = a$$

Another one: $\forall x, y \; yxxy = yx \implies (yxy = yx \vee (yxy)(yxx) = yx)$.

Tangent: tables

A finite subset H of a semigroup S = a partial multiplication table.

	a	b	$c = ab$	$d = aba$
a	$*$	c	a	$*$
b	$*$	$*$	$*$	$*$
c	d	$*$	$*$	$*$
d	$*$	$*$	d	$*$

This table does not have a *finite completion*.

Tangent: tables: finite completion

	a	b	c
a	b	$*$	$*$
b	$*$	c	$*$
c	$*$	$*$	a

$\rightarrow$

	a	a^2	a^4
a	a^2	$*$	$*$
a^2	$*$	a^4	$*$
a^4	$*$	$*$	a

$\rightarrow$

	a	a^2	a^3	a^4	a^5	a^6	a^7
a	$\mathbf{a^2}$	a^3	a^4	a^5	a^6	a^7	a
a^2	a^3	$\mathbf{a^4}$	a^5	a^6	a^7	a	a^2
a^3	a^4	a^5	a^6	a^7	a	a^2	a^3
a^4	a^5	a^6	a^7	$\mathbf{a}$	a^2	a^3	a^4
a^5	a^6	a^7	a	a^2	a^3	a^4	a^5
a^6	a^7	a	a^2	a^3	a^4	a^5	a^6
a^7	a	a^2	a^3	a^4	a^5	a^6	a^7

Tangent: tables: no completion

	a	b	c	d
a	a	b	c	d
b	b	a	$*$	$*$
c	c	$*$	b	b
d	d	$*$	b	b

Immdeiate reason

$bc = ddc = db = ddd = bd$, but $bbc = c \neq d = bbd$.

Properties of LEF - 1

Residually finite semigroup S

For every finite subset H of S there exists a finite (semi)group F_H and a homomorphism $\phi_H : S \to F_H$, such that ϕ_H separates elements of H.

Mutual position

S is residually finite $\implies$ S is LEF.
S is LEF and finitely presented $\implies$ S is residually finite.

Theorem

A finitely generated semigroup is LEF if and only if it is isomorphic to a direct limit of residually finite semigroups.

LEF is a genuinely different property from residual finiteness (and related concepts!)

Properties of LEF - 2

LEF is closed under...

- Taking subsemigroups;
- Adjoining 1 or 0;
- Direct products (sometimes it is enough to have one LEF term!);
- Wreath products with locally finite semigroups.

LEF is NOT closed under...

- Taking power semigroups;
- Reduction to direct factors (see above!)

Adding interactions

Assume that a semigroup S has a property X.

Expanding definition

For every finite subset H of a S there exists a finite (semi)group F_H with the property X and an injective map $f_H : H \to F_H$, such that for all $x, y \in H$ (with $xy \in H$) we have $(xy)f_H = (xf_H)(yf_H)$.

Driving question

If S is LEF and X, is it locally embeddable into the class of finite groups which are X?

Positive answers!

Proposition

A group is an LEF semigroup if and only if it is an LEF group.

Proposition

An inverse semigroup (i.e. a semigroup where for every x there exists a unique x^{-1} such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$) is an LEF semigroup if and only if it is locally embeddable into finite inverse semigroups.

Proposition

A Clifford semigroup (i.e. inverse semigroup where $xx^{-1} = x^{-1}x$) is an LEF semigroup if and only if it is locally embeddable into finite Clifford semigroups.

Negative answers!

Cancellativity

An example by A. Malcev, the cancellative semigroup $Sg\langle a, b, c, d, x, y, u, v | ax = by, cx = dy, au = bv\rangle$ is LEF, but it is not embeddable into finite cancellative semigroups, a.k.a. groups.

It follows from the fact that the relations would imply $cu = dv$ in a group.

$\mathcal{J}$-triviality

The semigroup $Sg\langle a, b, c, e, x | xb = cx, ac = ca, ea = ae, ec = ce, aex = ax, xca = xe\rangle$ is LEF, but it is not embeddable into finite $\mathcal{J}$-trivial semigroups.

It follows from the fact that in finite $\mathcal{J}$-trivial semigroups we would have $z^r = z^{r+1}$ for any z and some power r, while we also have $xa^n x = xa^{n+1} xb$.

Tangent: periodicity

Constructive proofs used only *periodicity* of finite semigroups.

Proposition

There exists a periodic structure which is not LEF.

Free Burnside groups

A group given by presentation $Gp\langle x_1, \ldots, x_m | R_r \rangle$ where $R_r = \{(w^r = 1) | w \in \{x_1^{\pm}, \ldots, x_m^{\pm}\}^*\}$;

These are infinite for large enough n. Also, they are not residually finite for large enough n.

Idea of proof

G - finitely presented group, contains $B(2, n)$. Even better, natural homomorphism $\pi : G \to G/G^n$ onto $B(2, n)$ is injective. Use the element inseparable from 1 and the word representing it to create a universal statement failed by G/G^n.

A "converse" property

LWF (semi)group S

For every finite subset H of S there exists a finite (semi)group D_H and a map $d_H : D_H \to S$, such that $H \subseteq D_H d_H$ and for all $x', y' \in D_H$ with $x'd_H, y'd_H \in H$ it holds that $(x'y')d_H = (x'd_H)(y'd_H)$.

Proposition

An LEF (semi)group is LWF.

Theorem (E. Gordon, A. Vershik, 1997)

A group is LEF if and only if it is LWF.

Proposition

There is an LWF semigroup which is not LEF.

An LWF example...

...and more!

The semigroups $Sg\langle a, b, c, d, e, x | abx = acx, ac = cd, ae = ed, xcd = xe, aex = ax, xexb = bxex\rangle$ and $Sg\langle a, b, c, e, x | abx = acx, ac = ca, ae = ea, xcd = xe, aex = ax, xexb = bxex\rangle$ are not LEF. However, they are $\mathcal{J}$-trivial and LWF.

The LWF property is not all-encompassing.

A usual suspect

The bicyclic monoid is not LWF.

Group-like connection

An inverse LWF semigroup which has finite number of idempotents is LEF.

Another "converse" property

LEF structures = substructures of the models of $Th(\mathcal{F})$, where $\mathcal{F}$ is the class of finite semigroups. These can be alternatively seen as ultraproducts of finite semigroups (UFS).

Second natural transformation

How can we characterise the structures which are *quotients* of UFS?

A preliminary proposition

A group is a quotient of an UFS if and only if it is a quotient of an ultraproduct of groups.

A non-example

The group $\mathbb{Z}$ is not isomorphic to a quotient of any UFS.

Open problems

- More LEF interactions, like E-unitary inverse semigroups (free inverse semigroup is locally embeddable into finite E-unitary semigroups) and completely regular semigroups (completely simple semigroups are locally embeddable into finite completely simple semigroups);
- Proving or disproving that inverse semigroups that are LWF are also LEF;
- Classifying partial multiplication tables;
- Delving into "semigroup soficity".

Thank you!