# Cartan semigroups

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# Cartan subalgebras (Renault 2008)

A pair $(A, B)$ of C*-algebras is a *Cartan pair* and say that $B$ is a *Cartan subalgebra* of $A$ if the following conditions hold:

1. $B$ is a maximal abelian subalgebra of $A$;
2. $B$ is regular in $A$, i.e., $\overline{\text{span}}(N(B)) = A$, where $N(B) = \{n \in A : nBn^* \cup n^*Bn \subseteq B\}$; and
3. there exists a faithful conditional expectation $E : A \twoheadrightarrow B$, i.e. $E : A \twoheadrightarrow B$ is contractive, completely positive s.t. $E|_B = id$ and $E(b_1 a b_2) = b_1 E(a) b_2$ for all $a \in A$ and $b_1, b_2 \in B$.

All Cartan pairs arise as *twisted groupoid C*-algebras* (of effective étale groupoids), and the groupoid and twist are unique.

A groupoid is a set $G$ with a map $(\alpha, \beta) \mapsto \alpha\beta : G^{(2)} \to G$, where $G^{(2)} \subset G \times G$ consists of composable pairs, and an inverse map $\gamma \mapsto \gamma^{-1} : G \to G$ such that:

- $(\gamma^{-1})^{-1} = \gamma$;
- $(\alpha, \beta), (\beta, \gamma) \in G^{(2)}$ implies $(\alpha\beta, \gamma), (\alpha, \beta\gamma) \in G^{(2)}$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
- $(\gamma^{-1}, \gamma) \in G^{(2)}$ and if $(\gamma, \beta) \in G^{(2)}$ then $\gamma^{-1}(\gamma\beta) = \beta$;
- $(\gamma, \gamma^{-1}) \in G^{(2)}$ and if $(\alpha, \gamma) \in G^{(2)}$ then $(\alpha\gamma)\gamma^{-1} = \alpha$.

For $\gamma \in G$, $\boldsymbol{s}(\gamma) = \gamma^{-1}\gamma$ is the source of $\gamma$ and $\boldsymbol{r}(\gamma) = \gamma\gamma^{-1}$ is the range of $\gamma$. The pair $(\alpha, \beta)$ is composable iff $\boldsymbol{r}(\beta) = \boldsymbol{s}(\alpha)$, and $G^{(0)} = \boldsymbol{r}(G) = \boldsymbol{s}(G)$ is the unit space of $G$.

A groupoid is called a topological groupoid if it has a topology for which the product and inversion are continuous.

Examples:

1. Every group is a groupoid (with the identity as its only unit).
2. Let $R$ be an equivalent relation on a set $X$. We have

$$R^{(2)} = \{((x,y),(y,z)) : (x,y),(y,z) \in R\},$$

and the product $(x,y)(y,z) = (x,z)$ and $(x,y)^{-1} = (y,x)$. Then,

$$\boldsymbol{r}(x,y) = (x,x) \text{ and } \boldsymbol{s}(x,y) = (y,y),$$

and the unit space of $R$ is the diagonal.

A subset $U$ of $G$ is called a bisection if $\exists$ an open subset $O$ of $G$ such that $U \subseteq O$ and $\boldsymbol{r}|_O, \boldsymbol{s}|_O$ are homeomorphisms onto open subsets of $G$.
We work with groupoids of following properties:

- $G$ is étale if $\boldsymbol{r}$ (or, equivalently, $\boldsymbol{s}$) is a local homeomorphism.
- $G$ is effective if $(\mathrm{Iso}(G))^\circ = G^{(0)}$.

Note that:

- If $G$ is étale, then $G^{(0)}$ is open; moreover, $G$ is étale if and only if $G$ has a basis of open bisections.
- Open bisections of an étale groupoid form an inverse semigroup.
- An étale groupoid is Hausdorff iff $G^{(0)}$ is closed.

## Definition

A Hausdorff topological groupoid $\Sigma$ is call a $\mathbb{T}$-groupoid if $\exists$ a free continuous action of $\mathbb{T}$ such that

$$t(ef) = (te)f = e(tf) \quad \text{for all } t \in \mathbb{T}, (e, f) \in \Sigma^{(2)}.$$

A twist is a groupoid homeomorphism $q : \Sigma \twoheadrightarrow G$ from a $\mathbb{T}$-groupoid $\Sigma$ onto a locally compact Hausdorff étale groupoid $G$ such that $\mathbb{T}$ acts transitively on each fibre, i.e.,

$$q^{-1}(\{q(e)\}) = \mathbb{T}e \text{ for all } e \in \Sigma.$$

- $\exists$ groupoid isomorphism $\iota : \mathbb{T} \times G^{(0)} \twoheadrightarrow q^{-1}(G^{(0)})$ such that

$$\iota(t, q(\boldsymbol{r}(e)))e = te = e\iota(t, q(\boldsymbol{s}(e))) \text{ for all } e \in \Sigma.$$

- If $q : \Sigma \twoheadrightarrow G$ is a twist then $\Sigma$ is locally compact.

Examples:

1. The Cartesian product groupoid $\mathbb{T} \times G$ is a twist over $G$, called the trivial twist.
2. Let $\sigma : G^{(2)} \to \mathbb{T}$ be a continuous normalised 2-cocycle. Then $\mathbb{T} \times_\sigma G$ is a twist over $G$, Namely, the set $\mathbb{T} \times G$ is endowed with the product topology, with multiplication given by

$$(z, \alpha)(w, \beta) := (\sigma(\alpha, \beta)zw, \alpha\beta),$$

and inversion by

$$(z, \alpha)^{-1} := (\overline{\sigma(\alpha, \alpha^{-1})z}, \alpha^{-1}).$$

Let $G$ be a locally compact Hausdorff étale groupoid and $\Sigma$ be a twist over $G$. We say that $f \in C(\Sigma)$ is $\mathbb{T}$-contravariant if

$$f(z \cdot \gamma) = \overline{z} \cdot f(\gamma) \quad \text{for all}\, \gamma \in \Sigma, z \in \mathbb{T}.$$

Define $C(\Sigma; G) := \{f \in C(\Sigma) : f \text{ is } \mathbb{T}\text{-contravariant}\}$ and let $C_c(\Sigma; G) := C(\Sigma; G) \cap C_c(\Sigma)$. Then $C_c(\Sigma; G)$ is a $*$-algebra under the operations

$$(f * g)(\gamma) := \sum_{\eta \in q(\gamma) G} f(\sigma(\eta)) g(\sigma(\eta)^{-1}\gamma) \quad \text{and} \quad f^*(\gamma) := \overline{f(\gamma^{-1})},$$

where $\sigma : G \to \Sigma$ is a section of $q$. Then the reduced twisted groupoid C*-algebra

$$C_r^*(\Sigma; G) := \overline{C_c(\Sigma; G)}^{\|\cdot\|_r},$$

where $\|f\|_r := \sup\{\|\pi_x(f)\| : x \in G^{(0)}\}$ and $\pi_x : C_c(\Sigma; G) \to B(L^2(G_x; \Sigma_x))$ is the regular rep. at $x \in G^{(0)}$.

**Theorem (Renault 2008, Raad 2022)**

*Let $\Sigma$ be a twist over an effective locally compact Hausdorff étale groupoid $G$. Then $(C_r^*(\Sigma; G), C_0(G^{(0)}))$ is a Cartan pair. Conversely, if $(A, B)$ is a Cartan pair, then there exists a twist $q : \Sigma \twoheadrightarrow G$, where $G$ is an effective locally compact Hausdorff étale groupoid such that $A \cong C_r^*(\Sigma; G)$ and $B \cong C_0(G^{(0)})$.*

Let $A$ be a C*-algebra and $N$ be a subset of $A$.

**Definition (Bice-Clark-L-McCormick)**

We call $N \subseteq A$ a Cartan semigroup if

1. $N$ is a $*$-subsemigroup of $A$ with dense span,
2. $B := C^*(N_+)$ is a commutative subsemigroup of $N$, where $N_+ := \{n^*n : n \in N\}$, and
3. there is a conditional expectation $E : A \twoheadrightarrow B$ such that

$$E(n)n^* \in B \text{ for all } n \in N.$$

In this case, we call $B$ the associated semi-Cartan subalgebra of $A$.

**Theorem (Bice-Clark-L-McCormick '25)**

*Let $A$ be a $C^*$-algebra containing a Cartan semigroup $N$ with associated semi-Cartan sublagebra $B$, and a stable expectation $E : A \twoheadrightarrow B$. Then $\exists$ an isomorphism*

$$\Psi : A \twoheadrightarrow C := \mathrm{cl}(C_c(\Sigma; G)), \text{ a twisted groupoid } C^*\text{-algebra},$$

*which maps $B$ onto $C_0(G^{(0)})$. Furthermore, if $E$ is faithful then*

$$C = \Psi(A) = C_r^*(\Sigma; G).$$

Note: $N \xrightarrow{\Psi} \{a \in C(\Sigma; G) : q(\mathrm{supp}^o) \text{ is a bisection}\}$ is a semigroup homomorphism.

## Lemma

- *Every semi-Cartan subalgebra $B$ contains an approximate unit for $A$.*
- $\mathbb{C}N = N \subseteq N(B)$.

Fix $(p_k)_k$, a sequence of nonzero polynomials, with zero constant terms that converge to 1 uniformly on all compact subsets of $\mathbb{R} \setminus \{0\}$. Then for any $a \in A$, it follows that

$$p_k(aa^*)a = ap_k(a^*a) \to a.$$

**Proposition**

*Suppose $(A, B)$ satisfies all properties of a Cartan pair except that the expectation $E$ is faithful. Then $N(B)$ forms a Cartan semigroup with associated semi-Cartan subalgebra $B = C^*(N(B)_+)$.*

To see that (3) holds: note that, for all $b \in B$ and $n \in N(B)$,

$$E(n)n^*nn^*b = E(n)n^*bnn^* = E(nn^*bn)n^* = E(bnn^*n)n^* = bE(n)n^*nn^*.$$

Likewise, $E(n)n^*(nn^*)^k b = bE(n)n^*(nn^*)^k$ for all $k > 1$, and hence

$$E(n)n^*b = \lim_k E(n)n^*p_k(nn^*)b = \lim_k bE(n)n^*p_k(nn^*) = bE(n)n^*.$$

As $B$ is a MASA, it follows that $E(n)n^* \in B$, showing that $N(B)$ is a Cartan semigroup.

## Definition (The restriction relation)

An element $m \in N$ is called a restriction of $n \in N$ if $\exists (b_k) \subseteq B$ such that

$$m = \lim_k mb_k = \lim_k nb_k,$$

and is denoted by $m \sqsubseteq n$.

- The restriction relation is a closed partial order relation on $N$.
- For any $m, n \in N$, if $m \sqsubseteq n$ then $n - m \sqsubseteq n$.
- For $n \in N$, $E(n) \sqsubseteq n$; in fact, $E(n) = \max\{b \in B : b \sqsubseteq n\}$.
- In twisted groupoid C*-algebra, it gives rise

$$m \sqsubseteq n \quad \Longleftrightarrow \quad j(m)|_{\mathrm{supp}^o(j(m))} = j(n)|_{\mathrm{supp}^o(j(m))}.$$

## Definition (Bice 2023)

For $m, n \in N$, the domination relation $<$ is defined by

$$m < n \iff \exists\, s \in N \;\text{ such that } m <_s n, \;\text{ where}$$
$$m <_s n \iff sm, ms, sn, ns \in B \text{ and } nsm = m = msn.$$

- $<$ is $E$-invariant, i.e. $m <_s n \Rightarrow E(m) <_{E(s)} E(n)$.
- $<$ is $*$-invariant, i.e. $m <_s n \Rightarrow m^* <_{s^*} n^*$.
- For $k, l, m, n \in N$, $k \sqsubseteq l < m \sqsubseteq n \Rightarrow k < n$.
- In twisted groupoid C*-algebra, it gives rise

$$m < n \Leftrightarrow \exists \text{ compact } K \text{ s.t. } \operatorname{supp}^o(j(m)) \subseteq K \subseteq \operatorname{supp}^o(j(n)).$$

Example

Take $G = \mathbb{Z}$, recall that $C_r^*(\mathbb{Z}) = C^*(\mathbb{Z}) \cong C(\mathbb{T})$ via Fourier transform. Then

- $N = \cup_{k\in\mathbb{Z}} \mathbb{C}\delta_k$ is a Cartan semigroup with
- $B := C^*(N_+) = \mathbb{C}1$, and
- expectation $E$ given by $E(a) = \varphi(a)1$, where $\varphi$ is any state on $C(\mathbb{T})$ such that $\varphi(\delta_k) = 0$ for all $0 \neq k \in \mathbb{Z}$.

However, $\mathbb{Z}$ is not effective, thus, $B$ is not a MASA (ABCCLMR, 2023).

Thank you for your attention!!


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