The quest for the clique numbers of commuting graphs of semigroups

Tânia Paulista

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Commuting graphs of semigroups

Let S be a finite non-commutative semigroup.

The **commuting graph** of S, denoted $\mathcal{G}(S)$, is the simple graph such that:

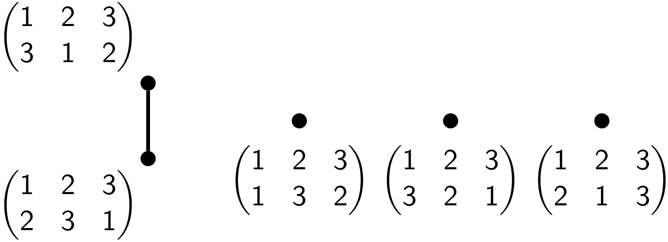
- $S \setminus Z(S)$ is the set of vertices, where

$$Z(S) = \{x \in S : xy = yx \text{ for all } y \in S\}.$$

- $\{x, y\}$ is an edge if $x \neq y$ and $xy = yx$.

Commuting graphs of semigroups: example

$\mathcal{G}(\mathcal{S}(\{1,2,3\}))$



(Full and partial) transformation semigroups

Let X be a finite set.

- **Full transformation semigroup** on X:

$$\mathcal{T}(X) = \{\,\text{Functions } \beta : \operatorname{Dom}\beta = X \text{ and } \operatorname{Im}\beta \subseteq X\,\}.$$

- **Partial transformation semigroup** on X:

$$\mathcal{P}(X) = \{\,\text{Functions } \beta : \operatorname{Dom}\beta \subseteq X \text{ and } \operatorname{Im}\beta \subseteq X\,\}.$$

Clique number

Let $G = (V, E)$ be a simple graph.

- A **clique** is a subset $K \subseteq V$ such that $\{u, v\} \in E$, for all distinct $u, v \in K$.
- The **clique number** of G, denoted $\omega(G)$, is the largest integer r such that G has a clique K such that $|K| = r$.

Clique number of a commuting graph

$\mathcal{G}(S)$		S
Cliques of maximum size C	$\longleftrightarrow$	Commutative subsemigroups of maximum size $C \cup Z(S)$

Lemma

Let S be a finite non-commutative semigroup. We have

$$\omega(\mathcal{G}(S)) = \begin{pmatrix} \textit{maximum size} \\ \textit{commutative} \\ \textit{subsemigroup of } S \end{pmatrix} - |Z(S)|.$$

Largest commutative subsemigroups

Let X be a finite set.

- Burns and Goldsmith (1989) characterized the largest abelian subgroups of the **symmetric group** $\mathcal{S}(X)$ on X.

$$\mathcal{S}(X) = \{\,\text{Functions } \beta : \beta \text{ is bijective and } \operatorname{Dom}\beta = X\,\}.$$

- Araújo, Bentz and Konieczny (2015) characterized the largest commutative subsemigroups of the **symmetric inverse semigroup** $\mathcal{I}(X)$ on X.

$$\mathcal{I}(X) = \{\,\text{Functions } \beta : \beta \text{ is injective and } \operatorname{Dom}\beta \subseteq X \text{ and } \operatorname{Im}\beta \subseteq X\,\}.$$

Largest commutative subsemigroups of idempotents

- For each $i \in X$ we define

$$\Gamma_i^X = \{\, \beta \in \mathcal{T}(X) : i\beta = i \text{ and } j\beta \in \{i, j\} \text{ for all } j \in X \setminus \{i\} \,\}.$$

- For all $i \in X$ we have $|\Gamma_i^X| = 2^{|X|-1}$.

Example

If $X = \{1, 2, 3\}$ and $i = 1$, then the elements of Γ_1^X are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Largest commutative subsemigroups of idempotents

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- For all $i \in X$ we have $|\Gamma_i^X| = 2^{|X|-1}$.

Theorem (P., 2025)

Let S be a commutative subsemigroup of idempotents of $\mathcal{T}(X)$. Then

- $|S| \leqslant 2^{|X|-1}$.
- $|S| = 2^{|X|-1} \iff (\exists\, i \in X)\ \ S = \Gamma_i^X$.

Largest commutative subsemigroups of idempotents

- **Symmetric inverse semigroup** on a set X:

$$\begin{aligned}\mathcal{I}(X) &= \{\,\text{Functions } \beta : \beta \text{ is injective and } \operatorname{Dom}\beta \subseteq X \text{ and } \operatorname{Im}\beta \subseteq X\,\} \\ &= \{\,\beta \in \mathcal{P}(X) : \beta \text{ is injective}\,\}.\end{aligned}$$

Corollary (P., 2025)

Let S be a commutative subsemigroup of idempotents of $\mathcal{P}(X)$. Then

- $|S| \leqslant 2^{|X|}$.
- $|S| = 2^{|X|} \iff S = E(\mathcal{I}(X))$.

Example

If $X = \{1, 2, 3\}$, then the elements of $E(\mathcal{I}(X))$ are

$$\emptyset \quad \begin{pmatrix}1\\1\end{pmatrix} \quad \begin{pmatrix}2\\2\end{pmatrix} \quad \begin{pmatrix}3\\3\end{pmatrix} \quad \begin{pmatrix}1&2\\1&2\end{pmatrix} \quad \begin{pmatrix}1&3\\1&3\end{pmatrix} \quad \begin{pmatrix}2&3\\2&3\end{pmatrix} \quad \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$$

Largest commutative subsemigroups

Theorem (P., 2025)

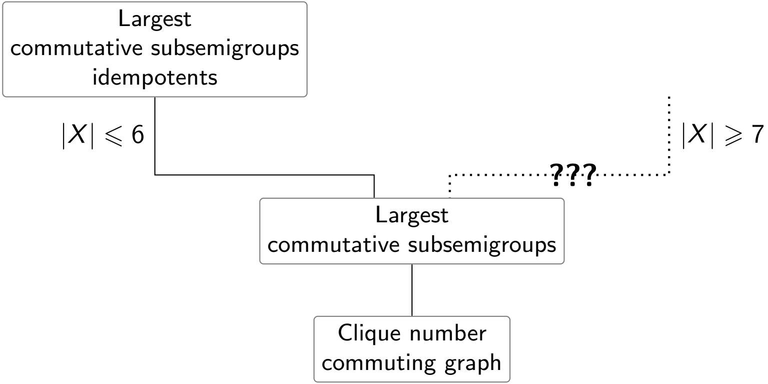
Suppose that $|X| \leqslant 6$. Let S be a commutative subsemigroup of $\mathcal{T}(X)$. Then

- $|S| \leqslant 2^{|X|-1}$.
- $|S| = 2^{|X|-1} \iff (\exists\, i \in X)\ S = \Gamma_i^X$ **or** $|X| = 2$ *and* $S \simeq C_2$.

Corollary (P., 2025)

Suppose that $|X| \leqslant 5$. Let S be a commutative subsemigroup of $\mathcal{P}(X)$. Then

- $|S| \leqslant 2^{|X|}$.
- $|S| = 2^{|X|} \iff S = E(\mathcal{I}(X))$.


Largest
commutative subsemigroups
idempotents
$|X| \leqslant 6$
$|X| \geqslant 7$
???
Largest
commutative subsemigroups
Clique number
commuting graph

Largest null subsemigroups

- **Null semigroup:** $S^2 = \{0\}$.
- $\xi(n) = \max\{t^{n-t} : t \in \{1, \dots, n\}\}$, $n \in \mathbb{N}$.
- $\alpha(n) = \max\{t \in \{1, \dots, n\} : t^{n-t} = \xi(n)\}$, $n \in \mathbb{N}$.
- If $x_1, \dots, x_{\alpha(|X|)} \in X$ are pairwise distinct, we define

$$N^X_{x_1,\dots,x_{\alpha(|X|)}} = \{\beta \in \mathcal{T}(X) : \{x_1, \dots, x_{\alpha(|X|)}\}\beta = \{x_1\} \text{ and } \operatorname{Im}\beta \subseteq \{x_1, \dots, x_{\alpha(|X|)}\}\}.$$

Example

If $X = \{1, 2, 3, 4\}$, then $\alpha(|X|) = 2$ and $\xi(|X|) = 4$ and the elements of $N^X_{1,2}$ are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

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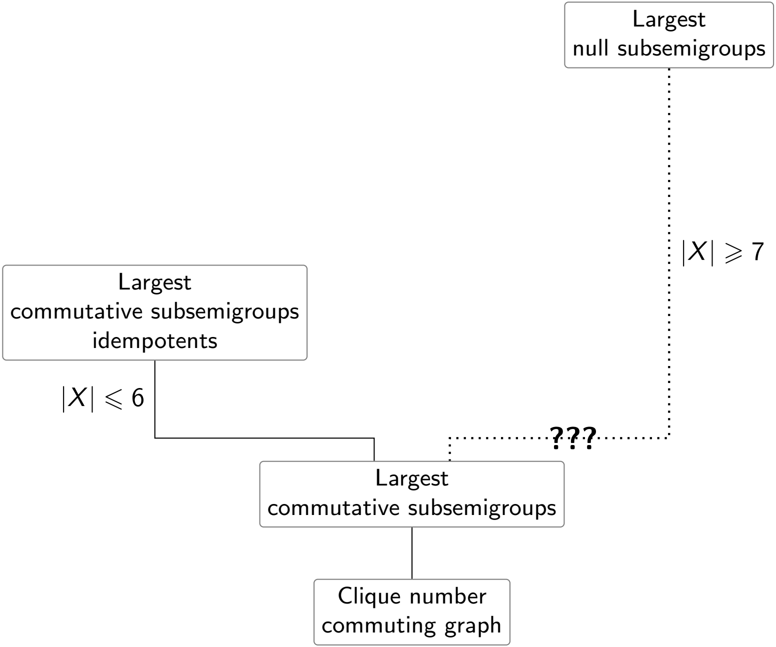
Theorem (Cameron et al., 2023)

Let S be a null subsemigroup of $\mathcal{T}(X)$. Then

- $|S| \leqslant \xi(|X|)$.
- $|S| = \xi(|X|) \iff (\exists\, x_1, \ldots, x_{\alpha(|X|)} \in X)\ S = N^X_{x_1,\ldots,x_{\alpha(|X|)}}$ **or** $|X| = 2$ *and* $S = \{\mathrm{id}_X\}$.


Largest
null subsemigroups
$|X| \geqslant 7$
Largest
commutative subsemigroups
idempotents
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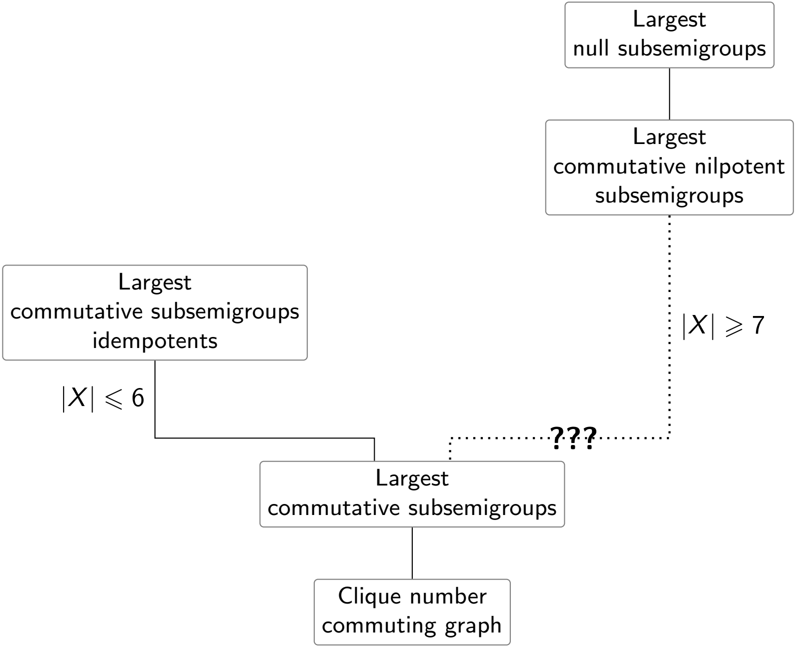
???
Largest
commutative subsemigroups
Clique number
commuting graph



Largest
null subsemigroups
Largest
commutative nilpotent
subsemigroups
Largest
commutative subsemigroups
idempotents
$|X| \geqslant 7$
$|X| \leqslant 6$
???
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Largest commutative nilpotent subsemigroups

- **Nilpotent semigroup:** $\exists m \in \mathbb{N} \quad S^m = \{0\}$.

Theorem (Cain, Malheiro, P., 2024)

Let S be a commutative nilpotent subsemigroup of $\mathcal{T}(X)$. Then

- $|S| \leqslant \xi(|X|)$.
- *If $|S| = \xi(|X|)$, then S is a null semigroup.*

Largest commutative nilpotent subsemigroups

- For each $B \subseteq X$ such that $|B| = \alpha(|X|+1) - 1$ we define

$$\Omega_B^X = \{\, \beta \in \mathcal{P}(X) : \operatorname{Dom}\beta \subseteq X \setminus B \text{ and } \operatorname{Im}\beta \subseteq B \,\}.$$

Example

If $X = \{1,2,3,4\}$, then $|B| = \alpha(|X|+1) - 1 = 2$. If $B = \{1,2\}$, then $X \setminus B = \{3,4\}$ and the elements of Ω_B^X are

Dom	$\emptyset$	$\{3\}$	$\{4\}$	$\{3,4\}$	
	$\emptyset$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$
		$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

Largest commutative nilpotent subsemigroups

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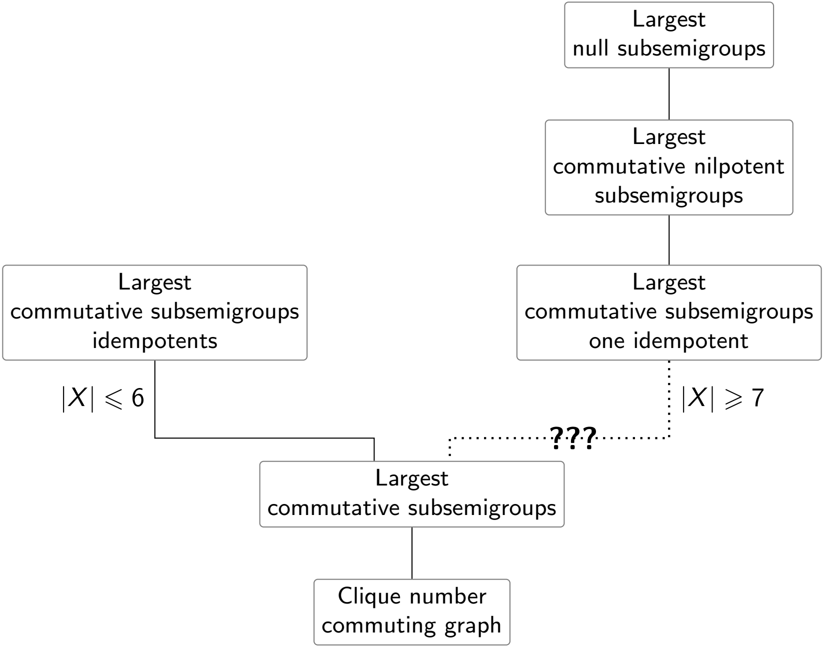
Corollary (P., 2025)

Let S be a commutative nilpotent subsemigroup of $\mathcal{P}(X)$. Then

- $|S| \leqslant \xi(|X|+1)$.
- $|S| = \xi(|X|+1) \iff |X| = 1$ *and* $S = \{\mathrm{id}_X\}$ **or** $S = \Omega_B^X$ *for some* $B \subseteq X$ *such that* $|B| = \alpha(|X|+1) - 1$.


Largest
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Largest
commutative subsemigroups
one idempotent
|X| ⩽ 6
|X| ⩾ 7
???
Largest
commutative subsemigroups
Clique number
commuting graph

Largest abelian subgroups

- $\mathcal{S}(X)$ is the **symmetric group** on X, formed by the bijections on X.

Theorem (Burns, Goldsmith, 1989)

Suppose that $|X| \geqslant 2$. *Then the maximum size of an abelian subgroup of* $\mathcal{S}(X)$ *is*

$$\begin{cases} 3^k & \text{if } |X| = 3k, \\ 4 \cdot 3^{k-1} & \text{if } |X| = 3k+1, \\ 2 \cdot 3^k & \text{if } |X| = 3k+2. \end{cases}$$

Moreover, the maximum-order abelian subgroups of $\mathcal{S}(X)$ *are isomorphic to*

$$\begin{cases} C_3^k & \text{if } |X| = 3k, \\ C_4 \times C_3^{k-1} \text{ or } C_2 \times C_2 \times C_3^{k-1} & \text{if } |X| = 3k+1, \\ C_2 \times C_3^k & \text{if } |X| = 3k+2. \end{cases}$$

Largest commutative subsemigroups with one idempotent

Theorem (P., 2025)

The maximum size of a commutative subsemigroup of $\mathcal{T}(X)$ with a unique idempotent is

$$\begin{cases} |X| & \text{if } |X| \leqslant 4, \\ \xi(|X|) & \text{if } |X| \geqslant 5. \end{cases}$$

Moreover, if S is a maximum-order commutative subsemigroup of $\mathcal{T}(X)$ with a unique idempotent, then

- *If $|X| \leqslant 3$, then S is a subgroup of $\mathcal{S}(X)$.*
- *If $|X| = 4$, then S is either a subgroup of $\mathcal{S}(X)$ or a null semigroup.*
- *If $|X| \geqslant 5$, then S is a null semigroup.*

Commutative semigroups with one idempotent

Lemma

Let S be a commutative subsemigroup of $\mathcal{T}(X)$ whose unique idempotent is e. Then

$$\{\beta|_{\operatorname{Im} e} : \beta \in S\}$$

is an abelian subgroup of $\mathcal{S}(\operatorname{Im} e)$.

$\operatorname{Im} e = X$

Theorem (Burns, Goldsmith, 1989)

The maximum size of an abelian subgroup of $\mathcal{S}(X)$ is

$$\begin{cases} 3^k & \text{if } |X| = 3k \\ 4 \cdot 3^{k-1} & \text{if } |X| = 3k+1 \\ 2 \cdot 3^k & \text{if } |X| = 3k+2. \end{cases}$$

Lemma

Let G be an abelian subgroup of $\mathcal{S}(X)$.

- *If $|X| \leqslant 4$, then $|G| \leqslant |X|$.*
- *If $|X| \geqslant 5$, then $|G| \leqslant \xi(|X|)$.*

$\operatorname{Im} e \neq X$

- $X = \{1, 2, 3, 4, 5, 6, 7\}$

- Commutative semigroup S with one idempotent:

$$\begin{pmatrix}1&2&3&4&5&6&7\\5&2&5&4&5&4&5\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6&7\\4&5&4&2&4&2&4\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5&6&7\\5&2&5&4&5&4&1\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6&7\\2&4&2&5&2&5&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5&6&7\\5&2&5&4&5&4&3\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6&7\\2&4&2&5&2&1&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5&6&7\\2&4&2&5&2&3&2\end{pmatrix}$$

S-partition

- $X = \{1, 2, 3, 4, 5, 6, 7\}$
- Commutative semigroup S with one idempotent:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{5} & \mathbf{2} & \mathbf{5} & \mathbf{4} & \mathbf{5} & \mathbf{4} & \mathbf{5} \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 4 & 2 & 4 & 2 & 4 \end{pmatrix} \qquad A_0 = \{2, 4, 5\}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 5 & 4 & 5 & 4 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 2 & 5 & 2 & 5 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 5 & 4 & 5 & 4 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 2 & 5 & 2 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 2 & 5 & 2 & 3 & 2 \end{pmatrix}$$

S-partition

- $X = \{1, 2, 3, 4, 5, 6, 7\}$
- Commutative semigroup S with one idempotent:

$$\begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & 6 & 7 \\ \mathbf{5} & 2 & \mathbf{5} & 4 & 5 & 4 & 5 \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & 6 & 7 \\ \mathbf{4} & 5 & \mathbf{4} & 2 & 4 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & 6 & 7 \\ \mathbf{5} & 2 & \mathbf{5} & 4 & 5 & 4 & 1 \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & 6 & 7 \\ \mathbf{2} & 4 & \mathbf{2} & 5 & 2 & 5 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & 6 & 7 \\ \mathbf{5} & 2 & \mathbf{5} & 4 & 5 & 4 & 3 \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & 6 & 7 \\ \mathbf{2} & 4 & \mathbf{2} & 5 & 2 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & 6 & 7 \\ \mathbf{2} & 4 & \mathbf{2} & 5 & 2 & 3 & 2 \end{pmatrix}$$

$A_0 = \{2, 4, 5\}$

$A_1 = \{1, 3\}$

S-partition

- $X = \{1, 2, 3, 4, 5, 6, 7\}$
- Commutative semigroup S with one idempotent:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \mathbf{6} & \mathbf{7} \\ 5 & 2 & 5 & 4 & 5 & \mathbf{4} & \mathbf{5} \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \mathbf{6} & \mathbf{7} \\ 4 & 5 & 4 & 2 & 4 & \mathbf{2} & \mathbf{4} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \mathbf{6} & \mathbf{7} \\ 5 & 2 & 5 & 4 & 5 & \mathbf{4} & \mathbf{1} \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \mathbf{6} & \mathbf{7} \\ 2 & 4 & 2 & 5 & 2 & \mathbf{5} & \mathbf{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \mathbf{6} & \mathbf{7} \\ 5 & 2 & 5 & 4 & 5 & \mathbf{4} & \mathbf{3} \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \mathbf{6} & \mathbf{7} \\ 2 & 4 & 2 & 5 & 2 & \mathbf{1} & \mathbf{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \mathbf{6} & \mathbf{7} \\ 2 & 4 & 2 & 5 & 2 & \mathbf{3} & \mathbf{2} \end{pmatrix}$$

$$A_0 = \{2, 4, 5\}$$
$$A_1 = \{1, 3\}$$
$$A_2 = \{6, 7\}$$

Property regarding S-partition

- Commutative semigroup S with one idempotent:

$$\begin{pmatrix}1&2&3&4&5&6&7\\ \mathbf{5}&2&\mathbf{5}&4&\mathbf{5}&4&5\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6&7\\ \mathbf{4}&5&\mathbf{4}&2&\mathbf{4}&2&4\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5&6&7\\ \mathbf{5}&2&\mathbf{5}&4&\mathbf{5}&4&1\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6&7\\ \mathbf{2}&4&\mathbf{2}&5&\mathbf{2}&5&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5&6&7\\ \mathbf{5}&2&\mathbf{5}&4&\mathbf{5}&4&3\end{pmatrix} \quad \begin{pmatrix}1&2&3&4&5&6&7\\ \mathbf{2}&4&\mathbf{2}&5&\mathbf{2}&1&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5&6&7\\ \mathbf{2}&4&\mathbf{2}&5&\mathbf{2}&3&2\end{pmatrix}$$

$$A_0 = \{2,4,5\}$$
$$A_1 = \{1,3\}$$
$$A_2 = \{6,7\}$$

Main Lemma

- $x \in A_i$
- $\alpha_1, \ldots, \alpha_m$ *agree on* $A_{<i}$

$\Longrightarrow$ $(x\alpha_k)\beta$ *are all equal.*

Words from transformations

- We order the elements of X in a way such that the elements of A_i appear before the elements of A_{i+1}, $i = 0, \ldots, k-1$.

$$A_0 = \{2, 4, 5\}$$
$$A_1 = \{1, 3\}$$
$$A_2 = \{6, 7\}$$

$$2, 4, 5, 1, 3, 6, 7$$

Words from transformations

- We obtain a set of words over X from the semigroup.

$$2, 4, 5, 1, 3, 6, 7$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 5 & 4 & 5 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 4 & 2 & 4 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 5 & 4 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 2 & 5 & 2 & 5 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 5 & 4 & 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 2 & 5 & 2 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 2 & 5 & 2 & 3 & 2 \end{pmatrix}$$

Words from transformations

- We obtain a set of words over X from the semigroup.

$$2, 4, 5, 1, 3, 6, 7$$

$$\begin{pmatrix}2 & 4 & 5 & 1 & 3 & 6 & 7\\ 2 & 4 & 5 & 5 & 5 & 4 & 5\end{pmatrix} \begin{pmatrix}2 & 4 & 5 & 1 & 3 & 6 & 7\\ 5 & 2 & 4 & 4 & 4 & 2 & 4\end{pmatrix}$$

$$\begin{pmatrix}2 & 4 & 5 & 1 & 3 & 6 & 7\\ 2 & 4 & 5 & 5 & 5 & 4 & 1\end{pmatrix} \begin{pmatrix}2 & 4 & 5 & 1 & 3 & 6 & 7\\ 4 & 5 & 2 & 2 & 2 & 5 & 2\end{pmatrix}$$

$$\begin{pmatrix}2 & 4 & 5 & 1 & 3 & 6 & 7\\ 2 & 4 & 5 & 5 & 5 & 4 & 3\end{pmatrix} \begin{pmatrix}2 & 4 & 5 & 1 & 3 & 6 & 7\\ 4 & 5 & 2 & 2 & 2 & 1 & 2\end{pmatrix}$$

$$\begin{pmatrix}2 & 4 & 5 & 1 & 3 & 6 & 7\\ 4 & 5 & 2 & 2 & 2 & 3 & 2\end{pmatrix}$$

$\longrightarrow$

2455545
2455541
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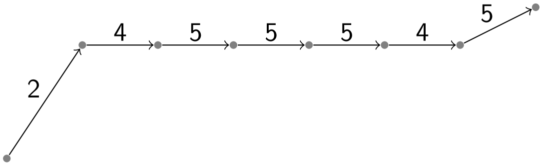
Tree from words



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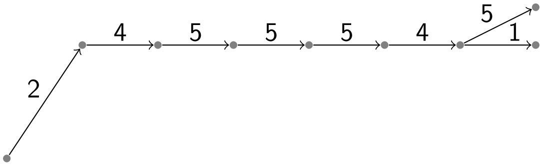
Tree from words



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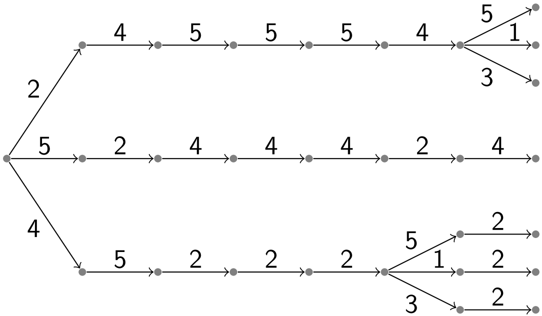
Tree from words



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2455543
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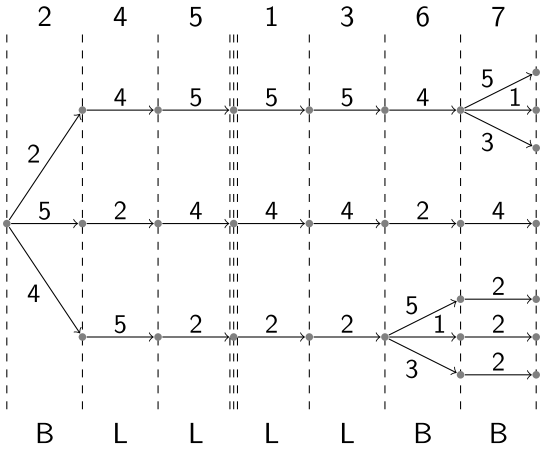
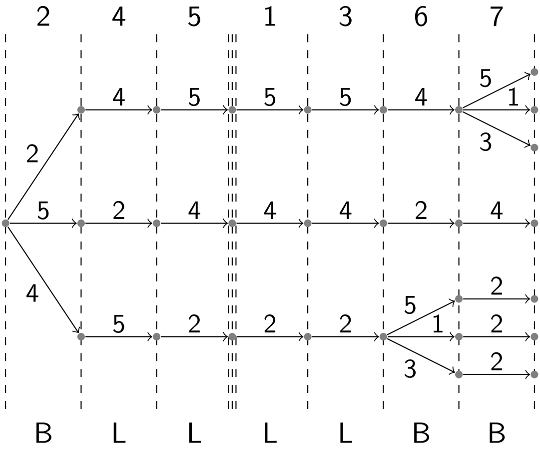
4522232

Tree of S



Property of the tree



Main Lemma

- $x \in A_i$
- $\alpha_1, \ldots, \alpha_m$ *agree on* $A_{<i}$

$\implies (x\alpha_k)\beta$ *are all equal.*

$A_0 = \{2, 4, 5\}$

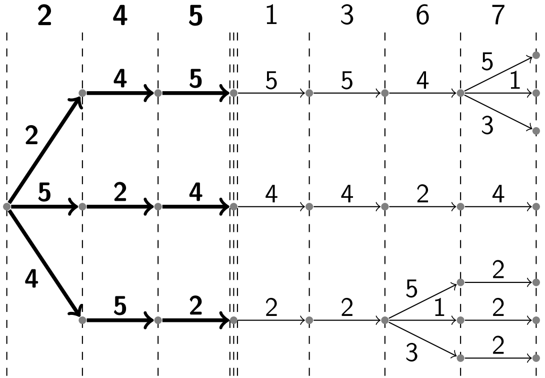
$A_1 = \{1, 3\}$

$A_2 = \{6, 7\}$

Tree of G

Abelian subgroup of $\mathcal{S}(\operatorname{Im} e) = \mathcal{S}(\{2,4,5\})$:

$$G = \{\beta|_{\operatorname{Im} e} : \beta \in S\} = \left\{ \begin{pmatrix} 2 & 4 & 5 \\ 2 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 5 \\ 5 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 5 \\ 4 & 5 & 2 \end{pmatrix} \right\}$$



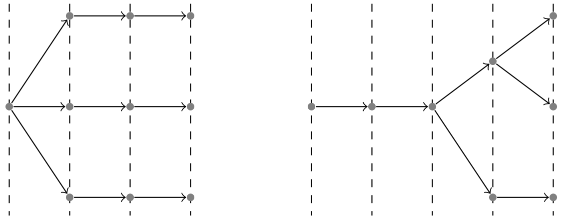
Modifying the tree of G

Lemma

Let G be an abelian subgroup of $\mathcal{S}(X)$. Then $|G| \leqslant \xi(|X|+1)$.

Null subsemigroup of $\mathcal{T}(\{2,4,5,6\})$ of size $|G| = 3$:

$$\begin{pmatrix} 2 & 4 & 5 & 6 \\ 2 & 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 & 5 & 6 \\ 2 & 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 4 & 5 & 6 \\ 2 & 2 & 4 & 2 \end{pmatrix}$$



Modifying the tree of G

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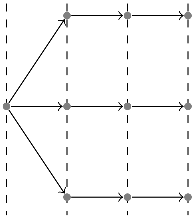
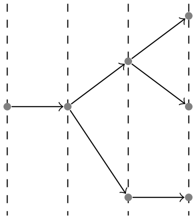
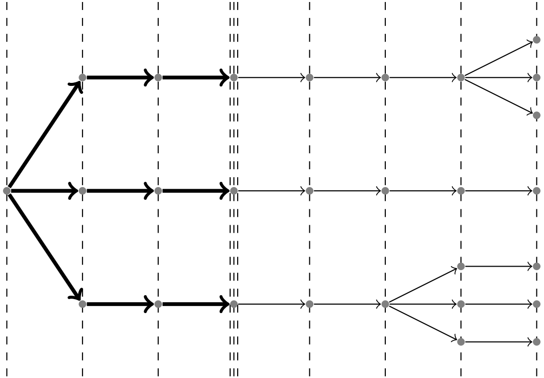
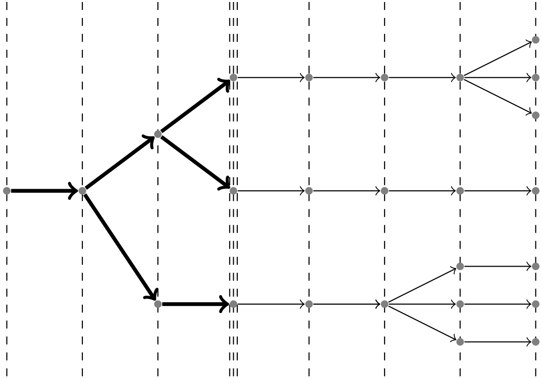




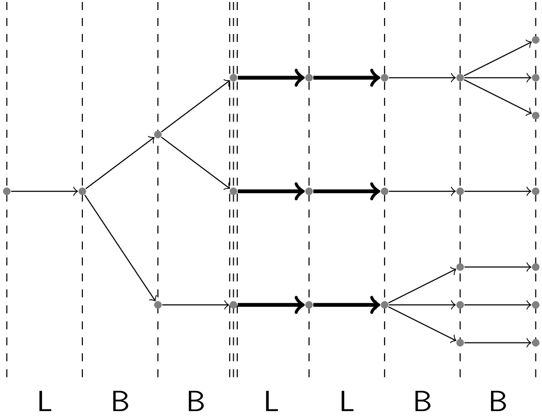
Modifying the tree of S



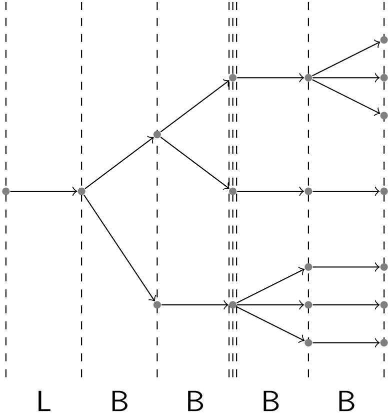
Modifying the tree of S



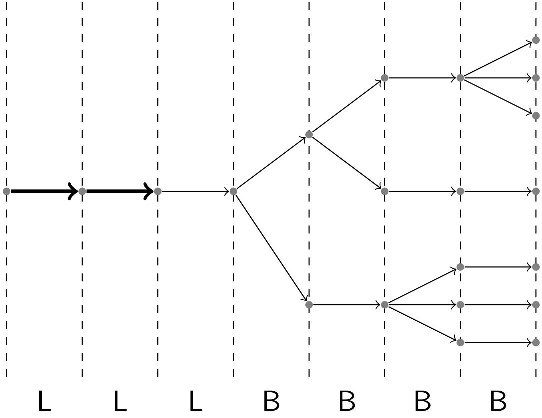
Modifying the tree of S



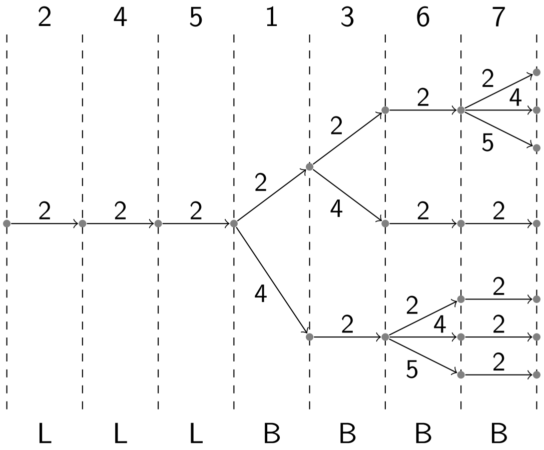
Modifying the tree of S



Modifying the tree of S



Modifying the tree of S



Transformations from words

- We obtain a null subsemigroup of $\mathcal{T}(X)$ from the set of words. The null semigroup has the same size as the initial commutative semigroup with one idempotent. So $|S| \leqslant \xi(|X|)$.

$$\begin{matrix} 2222222 \\ 2222224 \\ 2222225 \\ 2222422 \\ 2224222 \\ 2224242 \\ 2224252 \end{matrix} \longrightarrow \begin{matrix} \begin{pmatrix} 2 & 4 & 5 & 1 & 3 & 6 & 7 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 4 & 5 & 1 & 3 & 6 & 7 \\ 2 & 2 & 2 & 4 & 2 & 2 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 4 & 5 & 1 & 3 & 6 & 7 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix} & \begin{pmatrix} 2 & 4 & 5 & 1 & 3 & 6 & 7 \\ 2 & 2 & 2 & 4 & 2 & 4 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 4 & 5 & 1 & 3 & 6 & 7 \\ 2 & 2 & 2 & 2 & 2 & 2 & 5 \end{pmatrix} & \begin{pmatrix} 2 & 4 & 5 & 1 & 3 & 6 & 7 \\ 2 & 2 & 2 & 4 & 2 & 5 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 4 & 5 & 1 & 3 & 6 & 7 \\ 2 & 2 & 2 & 2 & 4 & 2 & 2 \end{pmatrix} & \end{matrix}$$

Commutative semigroups with one idempotent

Theorem (P., 2025)

The maximum size of a commutative subsemigroup of $\mathcal{T}(X)$ with a unique idempotent is

$$\begin{cases} |X| & \text{if } |X| \leqslant 4, \\ \xi(|X|) & \text{if } |X| \geqslant 5. \end{cases}$$

Moreover, if S is a maximum-order commutative subsemigroup of $\mathcal{T}(X)$ with a unique idempotent, then

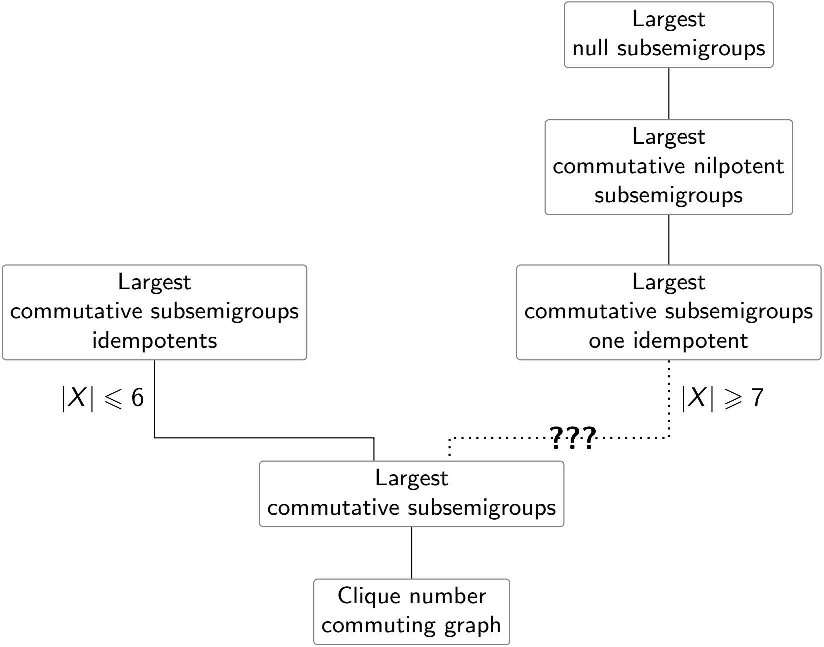
- *If $|X| \leqslant 3$, then S is a subgroup of $\mathcal{S}(X)$.*
- *If $|X| = 4$, then S is either a subgroup of $\mathcal{S}(X)$ or a null semigroup.*
- *If $|X| \geqslant 5$, then S is a null semigroup.*

Largest commutative subsemigroups with one idempotent

Corollary (P., 2025)

The maximum size of a commutative subsemigroup of $\mathcal{P}(X)$ with a unique idempotent is $\xi(|X|+1)$. Furthermore, if S is a maximum-order commutative subsemigroup of $\mathcal{P}(X)$ with a unique idempotent, then:

- *If $|X| \leqslant 2$, then either S is a subgroup of $\mathcal{S}(X)$ or S is a commutative nilpotent semigroup.*
- *If $|X| \geqslant 3$, then S is a commutative nilpotent semigroup.*


Largest
null subsemigroups
Largest
commutative nilpotent
subsemigroups
Largest
commutative subsemigroups
idempotents
Largest
commutative subsemigroups
one idempotent
$|X| \leqslant 6$
$|X| \geqslant 7$
???
Largest
commutative subsemigroups
Clique number
commuting graph

Largest commutative subsemigroups

Theorem (P., 2025)

Suppose that $|X| \geqslant 7$. Then

$$\begin{pmatrix} \text{maximum size} \\ \text{commutative} \\ \text{subsemigroup of } \mathcal{T}(X) \end{pmatrix} \geqslant \underbrace{\xi(|X|) + 1}_{N^X_{x_1,\ldots,x_{\alpha(|X|)}} \cup \{\mathrm{id}_X\}}.$$

Corollary (P., 2025)

Suppose that $|X| \geqslant 6$ and let Y be a set such that $|Y| = |X| + 1$. Then

$$\underbrace{\xi(|X|+1) + 1}_{\Omega^X_B \cup \{\mathrm{id}_X\}} \leqslant \begin{pmatrix} \text{maximum size} \\ \text{commutative} \\ \text{subsemigroup} \\ \text{of } \mathcal{P}(X) \end{pmatrix} \leqslant \begin{pmatrix} \text{maximum size} \\ \text{commutative} \\ \text{subsemigroup} \\ \text{of } \mathcal{T}(Y) \end{pmatrix}.$$

Clique numbers of commuting graphs

- $\omega(\mathcal{G}(S)) = \begin{pmatrix} \text{maximum size commutative} \\ \text{subsemigroup of } S \end{pmatrix} - |Z(S)|$.

Corollary (P., 2025)

Suppose that $|X| \geqslant 2$. Then

- *If $2 \leqslant |X| \leqslant 6$, then $\omega(\mathcal{G}(\mathcal{T}(X))) = 2^{|X|-1} - 1$.*
- *If $|X| \geqslant 7$, then $\omega(\mathcal{G}(\mathcal{T}(X))) \geqslant \xi(|X|)$.*

Corollary (P., 2025)

Suppose that $|X| \geqslant 2$. Then

- *If $2 \leqslant |X| \leqslant 5$, then $\omega(\mathcal{G}(\mathcal{P}(X))) = 2^{|X|} - 2$.*
- *If $|X| \geqslant 6$, then $\xi(|X|+1) - 1 \leqslant \omega(\mathcal{G}(\mathcal{P}(X))) \leqslant \omega(\mathcal{G}(\mathcal{T}(Y))) - 1$, where Y is a set such that $|Y| = |X| + 1$.*

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