# The Endomorphism Tower Problem

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## Endomorphisms

Let $S$ be a semigroup. An endomorphism of $S$ is a map $\phi$ from $S$ onto itself such that $x\phi\, y\phi = (xy)\phi$.

- The set of endomorphisms forms a monoid $\mathrm{End}(S)$.

## Automorphisms

An automorphism of $S$ is a bijective map $\psi$ from $S$ onto iself such that $x\psi\, y\psi = (xy)\psi$.

- The set of automorphisms forms a group $\mathrm{Aut}(S)$.

## Singular Endomorphism

An endomorphism that is not an automorphism is called a singular endomorphism.

- The set of singular endomorphisms forms a semigroup $\mathrm{SEnd}(S)$

## Aspects of Endomorphisms

Let $\phi$ be an endomorphism of $S$.

- The kernel of $\phi$ is the relation $\{(a, b) \in S \times S \mid a\phi = b\phi\}$
  - This is a congruence on $S$
  - If $S$ is a group we only need consider $\{a \in S \mid a\phi = 1\}$
- The image of $\phi$ is $\{x \in S \mid x = a\phi \text{ for some } a \in S\}$
- The cardinality of the image of $\phi$ is called its rank
- $\mathrm{End}(S)$ is the disjoint union of $\mathrm{Aut}(S)$ and $\mathrm{SEnd}(S)$

$$\mathrm{End}(S) = \mathrm{Aut}(S) \cup \mathrm{SEnd}(S)$$

- We define our maps piecewise

$$\phi : x \to \begin{cases} \text{im}(a) & \text{kernel-class of } a \\ \text{im}(b) & \text{kernel-class of } b \\ \dots & \dots \end{cases}$$

## A Simple Example

Let $M = \{1, e, f\}$ be the monoid of idempotents defined by

| | $1$ | $e$ | $f$ |
|---|---|---|---|
| $1$ | $1$ | $e$ | $f$ |
| $e$ | $e$ | $e$ | $f$ |
| $f$ | $f$ | $e$ | $f$ |

- $\{e, f\}$ is a right zero band (it is the minimal ideal of $M$)
- $M$ is $\{e, f\}$ with adjoined identity $1$.
- We can define the *natural order of idempotents* (n.p.o) as $i \leq j \iff ij = ji = i$.

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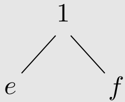
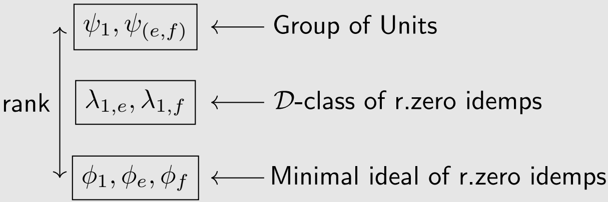



Then $\mathrm{End}(M)$ is $\{\psi_1, \psi_{(e\ f)}, \phi_{1,e}, \phi_{1,f}, \phi_e, \phi_f, \phi_1\}$.



## Example: Endomorphisms of $\mathcal{S}_{\mathrm{n}}$

Well known: $\mathrm{Aut}(\mathcal{S}_{\mathrm{n}}) = \{\psi_g : x \to g^{-1}xg \mid g \in \mathcal{S}_{\mathrm{n}}\}$
For singular endomorphisms look at kernel classes:

$$\{1\} \trianglelefteq \mathcal{A}_{\mathrm{n}} \trianglelefteq \mathcal{S}_{\mathrm{n}}$$

1. Automorphisms: $\psi_g : x \to g^{-1}xg$
2. Singular Endomorphisms:

$$\phi_g : x \to \begin{cases} g & \text{if } x \in \mathcal{S}_{\mathrm{n}} \setminus \mathcal{A}_{\mathrm{n}} \\ 1 & \text{if } x \in \mathcal{A}_{\mathrm{n}} \end{cases}$$

   where $g^2 = 1$.

## Full Transformation Monoid (Schein and Teclezghi)

Possible kernel classes (congruences):

$$id \subseteq R_{\mathcal{A}_{\mathrm{n}}} \subseteq R_{\mathcal{S}_{\mathrm{n}}} \subseteq \mathcal{T}_{\mathrm{n}} \times \mathcal{T}_{\mathrm{n}}$$

1. Automorphisms: $\psi_g : x \to g^{-1}xg$ where $g \in \mathcal{S}_{\mathrm{n}}$
2. Singular Endomorphisms:

$$\phi_{t,e} : x \to \begin{cases} t & \text{if } x \in \mathcal{S}_{\mathrm{n}} \setminus \mathcal{A}_{\mathrm{n}} \\ t^2 & \text{if } x \in \mathcal{A}_{\mathrm{n}} \\ e & \text{if } x \in \mathcal{T}_n \setminus \mathcal{S}_{\mathrm{n}} \end{cases}$$

where $t^3 = t$ and $te = et = e = e^2$.

## Full Transformation Monoid (Schein and Teclezghi)

Automorphisms: $\psi_g : x \to g^{-1}xg$ where $g \in \mathcal{S}_\mathrm{n}$
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where $t^3 = t$ and $te = et = e = e^2$.

## Partition Monoid (Mazorchuk)

$$\psi_g : x \to g^{-1}xg$$

$$\phi_{t,e} : x \to \begin{cases} t & x \in \mathcal{S}_\mathrm{n} \setminus \mathcal{A}_\mathrm{n} \\ t^2 & x \in \mathcal{A}_\mathrm{n} \\ e & \text{otw} \end{cases}$$

where $t^3 = t$ and $te = et = e$.

$$\lambda_{g,e} : x \to \begin{cases} g^{-1}xg & \text{if } x \in \mathcal{S}_\mathrm{n} \\ e & \text{otw} \end{cases}$$

where $ge = eg = e$ for all $g$.

# The Structure of $\mathrm{End}(\mathcal{S}_\mathrm{n})$

Elements:

$$\psi_g : x \to g^{-1}xg \qquad (g \in \mathcal{S}_\mathrm{n})$$

$$\phi_g : x \to \begin{cases} g & \text{if } x \in \mathcal{S}_\mathrm{n} \setminus \mathcal{A}_\mathrm{n} \\ 1 & \text{if } x \in \mathcal{A}_\mathrm{n} \end{cases} \qquad (g \in \mathcal{S}_\mathrm{n},\ g^2 = 1)$$

Cayley table of $\mathrm{End}(\mathcal{S}_\mathrm{n})$:

| | $\psi_h$ | $\phi_h$ |
|---|---|---|
| $\psi_g$ | $\psi_{gh}$ | $\phi_h$ |
| $\phi_g$ | $\phi_{h^{-1}gh}$ | $(*)$ |

$$(*) \quad \phi_g\phi_h = \begin{cases} \phi_h & \text{if } g \in \mathcal{S}_\mathrm{n} \setminus \mathcal{A}_\mathrm{n} \\ \phi_1 & \text{if } g \in \mathcal{A}_\mathrm{n} \end{cases}$$

(So $\phi_g^{\,2} = \phi_g$ if $g \in \mathcal{S}_\mathrm{n} \setminus \mathcal{A}_\mathrm{n}$ and $\phi_g^{\,2} = \phi_1$ if $g \in \mathcal{A}_\mathrm{n}$)

## The Structure of $\mathrm{End}(\mathcal{S}_\mathrm{n})$

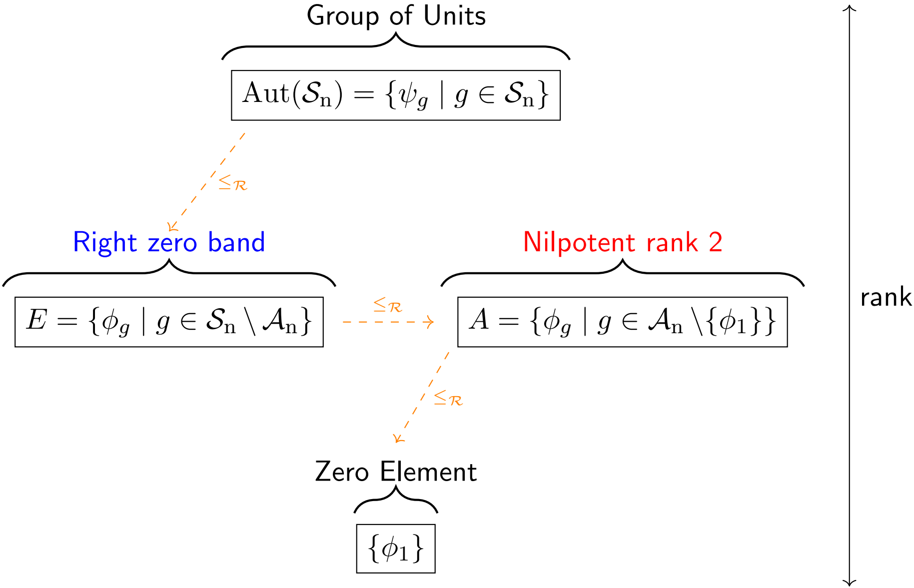
Elements:

$$\psi_g : x \to g^{-1}xg \qquad (g \in \mathcal{S}_\mathrm{n})$$

$$\phi_g : x \to \begin{cases} g & \text{if } x \in \mathcal{S}_\mathrm{n} \setminus \mathcal{A}_\mathrm{n} \\ 1 & \text{if } x \in \mathcal{A}_\mathrm{n} \end{cases} \qquad (g \in \mathcal{S}_\mathrm{n},\ g^2 = 1)$$

Separate by Rank and Type:

- Units: $\{\psi_g \mid g \in \mathcal{S}_\mathrm{n}\}$
- Rank 2 idempotents $\{\phi_g \mid g \in \mathcal{S}_\mathrm{n} \setminus \mathcal{A}_\mathrm{n}\}$
- Rank 2 nilpotent $\{\phi_h \mid g \in \mathcal{A}_\mathrm{n}\}$
- The zero $\phi_1$


Group of Units
$\mathrm{Aut}(\mathcal{S}_\mathrm{n}) = \{\psi_g \mid g \in \mathcal{S}_\mathrm{n}\}$
$\leq_\mathcal{R}$
Right zero band
$E = \{\phi_g \mid g \in \mathcal{S}_\mathrm{n} \setminus \mathcal{A}_\mathrm{n}\}$
$\leq_\mathcal{R}$
Nilpotent rank 2
$A = \{\phi_g \mid g \in \mathcal{A}_\mathrm{n} \setminus \{\phi_1\}\}$
$\leq_\mathcal{R}$
Zero Element
$\{\phi_1\}$
rank

## $\mathcal{T}_n$ and $\mathcal{P}_X$ are similar

<table>
<tr><td colspan="4">$\mathrm{Aut}(\mathcal{T}_n) \cong \mathcal{S}_n$</td></tr>
<tr><td>$E_3$</td><td colspan="2">$A$</td><td>$B$</td></tr>
<tr><td colspan="2">$E_2$</td><td colspan="2">$C$</td></tr>
<tr><td colspan="4">$E_1$</td></tr>
</table>

Table: Partiton of $\mathrm{End}(\mathcal{T}_n)$ by rank and type

<table>
<tr><td colspan="4">$\mathrm{Aut}(\mathcal{P}_X) \cong \mathcal{S}_n$</td></tr>
<tr><td colspan="4">$Z$</td></tr>
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</table>

Table: Partition of $\mathrm{End}(\mathcal{P}_X)$ by rank and type.

## Automorphism Towers

Construct a sequence (a tower): Let $G$ a group

$$G, \quad \mathrm{Aut}(G), \quad \mathrm{Aut}(\mathrm{Aut}(G)), \quad \mathrm{Aut}(\mathrm{Aut}(\mathrm{Aut}(G))), \quad \dots$$

Simplify notation:

$$G, \quad \mathrm{Aut}_1(G), \quad \mathrm{Aut}_2(G), \quad \mathrm{Aut}_3(G), \quad \dots$$

**The Automorphism Tower Problem:**

Does there exist $i$ such that $\mathrm{Aut}_i(G) \cong \mathrm{Aut}_{i+1}(G)$?

(Does the sequence terminate up to isomorphism?)

### Wielandt, 1939

The automorphism tower of any centerless finite group terminates in finitely many steps.

### Hulse, 1970

The automorphism tower of any centerless polycyclic group terminates in a countable number of steps.

### Thomas, 1985

The automorphism tower of any centerless group terminates.

### Hamkins, 1998

Every group has a terminating automorphism tower.

## Endomorphism Towers

Construct a sequence (the tower): Let $M$ a monoid

$$M, \quad \mathrm{End}(M), \quad \mathrm{End}(\mathrm{End}(M)), \quad \mathrm{End}(\mathrm{End}(\mathrm{End}(M))), \quad \ldots$$

Simplify notation:

$$M, \quad \mathrm{End}_1(M), \quad \mathrm{End}_2(M), \quad \mathrm{End}_3(M), \quad \ldots$$

**The Endomorphism Tower Problem:**

Does there exist $i$ such that $\mathrm{End}_i(M) \cong \mathrm{End}_{i+1}(M)$?

(Does the sequence terminate up to isomorphism?)

## Theorem

The only finite monoid with a terminating endomorphism tower is the trivial monoid $M = \{1\}$.

**Proof.** Let $e \in M$ be idempotent. Then the map

$$\phi_e : x \to e \quad (\forall x \in M)$$

is an endomorphism. Moreover, $\phi_e$ is an idempotent. Thus,

$$|E(M)| \leq |E(\mathrm{End}(M))|.$$

Further, if $M$ is non-trivial then the identity map is not $\phi_e$ for any $e$. Thus,

$$|E(M)| < |E(\mathrm{End}(M))|.$$

So $M \cong \mathrm{End}(M)$ if the identity is a constant map. I.e. $\mathrm{End}(M) = \{\phi_1\}$.

## Theorem

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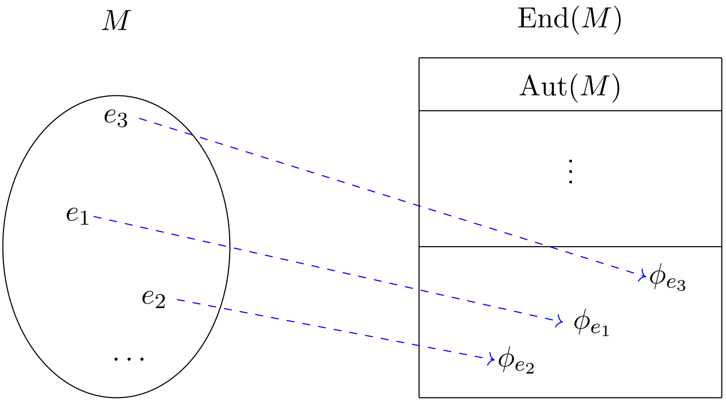

## Tower of $\mathcal{S}_\mathrm{n}$ - The First Level

We had elements $\psi_g$ (automorphisms) and $\phi_g$ (rank $\leq 2$ singular endomorphisms).

<table>
<tr><td colspan="2">$\mathrm{Aut}(\mathcal{S}_\mathrm{n}) \cong \mathcal{S}_\mathrm{n}$</td></tr>
<tr><td>$E$</td><td>$A$</td></tr>
<tr><td colspan="2">$\phi_1$</td></tr>
</table>

Notice

- Units of $\mathrm{End}(\mathcal{S}_\mathrm{n})$ is isomorphic to Units of $\mathcal{S}_\mathrm{n}$
- $\mathcal{S}_\mathrm{n}$ embeds into $\mathrm{End}(\mathcal{S}_\mathrm{n})$

So what about $\mathrm{End}_2(\mathcal{S}_\mathrm{n})$?

# Tower of $S_n$ - The Second Level

Cayley Table of $\mathrm{End}_2(\mathcal{S}_\mathrm{n})$

| | $\Psi_g$ | $\Xi^g_{M(X)}$ | $\Delta^g_{M(X)}$ | $\Phi^g_S$ | $\Omega_{1,g}$ | $\Omega_{2,g}$ | $\Omega_{3,g}$ |
|---|---|---|---|---|---|---|---|
| $\Psi_h$ | $\Psi_{hg}$ | $\Xi^{hg}_{M(X)}$ | $\Delta^{hg}_{M(X)}$ | $\Phi^g_S$ | $\Omega_{1,g}$ | $\Omega_{2,g}$ | $\Omega_{3,g}$ |
| $\Xi^h_{M(Y)}$ | $\Xi^{hg}_{M(Y)}$ | $\Xi^{hg}_{M(Y\cup X)}$ | (1) | $\Phi^g_{M(Y)S}$ | $\Omega_{1,g}$ | $\Omega_{2,g}$ | $\Omega_{3,g}$ |
| $\Delta^h_{M(Y)}$ | $\Delta^{hg}_{M(Y)}$ | (2) | $\Delta^{hg}_{M(Y\cup X)}$ | $\Phi^g_{M(Y)S}$ | (3) | (4) | $\Omega_{3,g}$ |
| $\Phi^h_T$ | $\Phi^{h^g}_{T\Psi_g}$ | $\Phi^{h^g}_{TM(X)\Psi_g}$ | $\Phi^{h^g}_{TM(X)\Psi_g}$ | (5) | (6) | (7) | $\Omega_{3,g}$ |
| $\Omega_{1,h}$ | $\Omega_{1,h^g}$ | (8) | $\Omega_{1,h^g}$ | $\Omega_{1,hS}$ | (9) | (10) | $\Omega_{2,g}$ |
| $\Omega_{2,h}$ | $\Omega_{2,h^g}$ | (11) | (12) | (13) | $\Omega_{1,g}$ | $\Omega_{2,g}$ | $\Gamma_{\phi_1}$ |
| $\Omega_{3,h}$ | $\Omega_{3,h^g}$ | (14) | (15) | (16) | $\Phi^1_{C_g}$ | $\Omega_{3,g}$ | $\Gamma_{\phi_1}$ |

## Tower of $\mathcal{S}_n$ - The Second Level

| | | |
|---|---|---|
| $\Xi^g_{M(X)}$ | $g \in \mathcal{S}_n$ and $X$ is empty or a union of conjugacy classes in $A \cup \{\phi_1\}$ | injective on $\mathrm{Aut}(\mathcal{S}_n)$ and only exists when $n$ is divisible by 4 |
| $\Delta^g_{M(X)}$ | $g \in \mathcal{S}_n$, $X$ is $\mathrm{SEnd}(\mathcal{S}_n$ or a union of conjugacy classes in $A \cup \{\phi_1\}$ intersecting $A$ | injective on $\mathrm{Aut}(\mathcal{S}_n)$ |
| $\Phi^h_S$ | $h \in \mathcal{S}_n$ with $h^2 = 1$, $S$ preserves rank & type and kernel classes are unions of conjugacy classes | rank 2 on $\mathrm{Aut}(\mathcal{S}_n)$ forming a subgroup of $\mathrm{Aut}(\mathcal{S}_n)$, and for all $\phi_t \in \mathrm{im}(S)$ we have $t^h = t$ |
| $\Omega_{1,g}$ | $\phi_g \in E \cup \{\phi_1\}$ | $\omega \mapsto \psi_1$ if $\omega \in \mathrm{Aut}(\mathcal{S}_n) \cup E$ and $\omega \mapsto \phi_g$ if $\omega \in A \cup \{\phi_1\}$ |
| $\Omega_{2,g}$ | $\phi_g \in E$ | $\omega \mapsto \phi_g$ if $\omega \in \mathrm{Aut}(\mathcal{S}_n) \cup E$ and $\omega \mapsto \phi_1$ if $\omega \in A \cup \{\phi_1\}$ |
| $\Omega_{3,g}$ | $\phi_g \in E$ | $\omega \mapsto \phi_g$ if $\omega \in \mathrm{Aut}(\mathcal{S}_n)$ and $\omega \mapsto \phi_1$ if $\omega \in \mathrm{SEnd}(\mathcal{S}_n)$ |
| $\Gamma_\lambda$ | $\lambda^2 = \lambda \in \mathrm{End}(\mathcal{S}_n)$ | constant maps to $\lambda$ |

# Tower of $S_n$ - The Group of Units

$\longrightarrow$

<table>
<tr><td colspan="3">$\Psi_g$</td></tr>
<tr><td colspan="2">$\Delta^g_{M(X)}$</td><td>$\Xi^g_{M(X)}$</td></tr>
<tr><td colspan="2">$\Phi^g_S$</td><td>$\Phi^k_T$</td></tr>
<tr><td colspan="3">$\Phi^1_R$</td></tr>
<tr><td>$\Omega_1$</td><td>$\Omega_2$</td><td>$\Omega_3$</td></tr>
<tr><td colspan="3">$\Gamma$</td></tr>
</table>

$\longleftarrow$

Group of Units of $\mathrm{End}_2(\mathcal{S}_\mathrm{n})$:

$$\Psi_g : \omega \to \psi_g^{-1} \omega \psi_g$$

# Tower of $S_n$ - The Group of Units

Cayley Table of $\mathrm{End}_2(\mathcal{S}_\mathrm{n})$

| | $\Psi_g$ | $\Xi^g_{M(X)}$ | $\Delta^g_{M(X)}$ | $\Phi^g_S$ | $\Omega_{1,g}$ | $\Omega_{2,g}$ | $\Omega_{3,g}$ |
|---|---|---|---|---|---|---|---|
| $\Psi_h$ | $\Psi_{hg}$ | $\Xi^{hg}_{M(X)}$ | $\Delta^{hg}_{M(X)}$ | $\Phi^g_S$ | $\Omega_{1,g}$ | $\Omega_{2,g}$ | $\Omega_{3,g}$ |
| $\Xi^h_{M(Y)}$ | $\Xi^{hg}_{M(Y)}$ | $\Xi^{hg}_{M(Y\cup X)}$ | (1) | $\Phi^g_{M(Y)S}$ | $\Omega_{1,g}$ | $\Omega_{2,g}$ | $\Omega_{3,g}$ |
| $\Delta^h_{M(Y)}$ | $\Delta^{hg}_{M(Y)}$ | (2) | $\Delta^{hg}_{M(Y\cup X)}$ | $\Phi^g_{M(Y)S}$ | (3) | (4) | $\Omega_{3,g}$ |
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## Tower of $S_n$ - The Group of Units

<table>
<tr><td colspan="3">$\Psi_g$</td></tr>
<tr><td colspan="2">$\Phi_S^g$</td><td>$\Phi_T^k$</td></tr>
<tr><td colspan="3">$\Phi_R^1$</td></tr>
<tr><td>$\Omega_1$</td><td>$\Omega_2$</td><td>$\Omega_3$</td></tr>
<tr><td colspan="3">$\Gamma$</td></tr>
</table>

Group of Units of $\mathrm{End}_2(\mathcal{S}_\mathrm{n})$:

- Elements: $\Psi_g : \omega \to \psi_g^{-1}\omega\psi_g$,
- Structure: $\cong \mathcal{S}_\mathrm{n}$.

| Monoid | $\mathcal{S}_\mathrm{n}$ | $\mathrm{End}(\mathcal{S}_\mathrm{n})$ | $\mathrm{End}_2(\mathcal{S}_\mathrm{n})$ | $\mathrm{End}_3(\mathcal{S}_\mathrm{n})$ | $\mathrm{End}_4(\mathcal{S}_\mathrm{n})$ |
|---|---|---|---|---|---|
| Group of Units | $\mathcal{S}_\mathrm{n}$ | $\cong \mathcal{S}_\mathrm{n}$ | $\cong \mathcal{S}_\mathrm{n}$ | $\cong \mathcal{S}_\mathrm{n}$ | $\cong \mathcal{S}_\mathrm{n} \times K$ |

where $K$ is hopefully a trivial group.

## Group of Units - The General Idea

We have a monoid $M$ with group of units $\mathcal{S}_{\mathrm{n}}$.

- Automorphisms of $\mathcal{S}_{\mathrm{n}}$: $\psi_g : x \to g^{-1}xg$
- $\psi_g$ can also be used as an automorphism of $M$.

$$\mathrm{Aut}(\mathcal{S}_{\mathrm{n}}) \leq \mathrm{Aut}(M)$$

- $\mathcal{S}_{\mathrm{n}}$ must be mapped to itself by any autom of $M$.

$$\forall \lambda \in \mathrm{Aut}(M), \quad \lambda|_{\mathcal{S}_{\mathrm{n}}} \in \mathrm{Aut}(\mathcal{S}_{\mathrm{n}})$$

That is, if $\lambda$ is in $\mathrm{Aut}(M)$ then there is some $\psi_g$ such that $\lambda\psi_g$ is the identity on the units $\mathcal{S}_{\mathrm{n}}$.

## Groups of Units - The General Idea

- Groups: $G, K, H$
- Injective morphism $\mathfrak{i} : K \to G$
- Surjective morphism $\mathfrak{r} : G \to H$
- $\mathrm{im}(\mathfrak{i}) = \ker(\mathfrak{r})$

Then the following is a short exact sequence:

$$K \xrightarrow{\mathfrak{i}} G \xrightarrow{\mathfrak{r}} H.$$

If there is $\mathfrak{e} : H \to G$ such that $\mathfrak{e}\mathfrak{r} = id_H$ then the sequence *splits*.

$$G \cong K \rtimes H.$$

## Groups of Units - The General Idea

For any monoid $M$ with units $\mathcal{S}_{\mathrm{n}}$ let:

- $G = \mathrm{Aut}(M)$
- $H = \mathrm{Aut}(\mathcal{S}_{\mathrm{n}})$
- $\mathfrak{r} : \mathrm{Aut}(M) \to \mathrm{Aut}(\mathcal{S}_{\mathrm{n}})$ be the restriction map $\lambda\mathfrak{r} = \lambda|_{\mathcal{S}_{\mathrm{n}}}$
- $K = \ker(\mathfrak{r})$
- $\mathfrak{i} : \ker(\mathfrak{r}) \to \mathrm{Aut}(M)$ be the inclusion map.

Then

$$\ker(\mathfrak{r}) \xrightarrow{\mathfrak{i}} \mathrm{Aut}(M) \xrightarrow{\mathfrak{r}} \mathrm{Aut}(\mathcal{S}_{\mathrm{n}})$$

is a short exact sequence. We split this sequence with the map

$$\psi_g : \mathcal{S}_{\mathrm{n}} \to \mathcal{S}_{\mathrm{n}} \quad \mapsto \quad \psi_g : M \to M$$

Yielding

$$\mathrm{Aut}(M) \cong \ker(\mathfrak{r}) \rtimes \mathrm{Aut}(\mathcal{S}_{\mathrm{n}}).$$

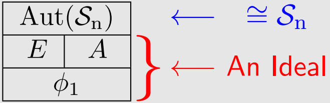
## Tower of $\mathcal{S}_n$ - Embeddings

**Recall:** $\mathcal{S}_n \hookrightarrow \mathrm{End}(\mathcal{S}_n)$.

**Q.** Does $\mathrm{End}(\mathcal{S}_n)$ embed into its endomorphism monoid?

I.e: find maps in $\mathrm{End}_2(\mathcal{S}_n)$ that 'look like' the elements of $\mathrm{End}(\mathcal{S}_n)$.

**Structure of** $\mathrm{End}(\mathcal{S}_n)$**:**




- $\psi_g : x \to g^{-1}xg \quad (g \in \mathcal{S}_n)$
- $\phi_g : x \to \begin{cases} g & \text{if } x \in \mathcal{S}_n \setminus \mathcal{A}_n \\ 1 & \text{if } x \in \mathcal{A}_n \end{cases}$

  for $g^2 = 1$.

# Tower of $\mathcal{S}_{\mathrm{n}}$ - Embeddings

**Structure of** $\mathrm{End}(\mathcal{S}_{\mathrm{n}})$**:**

| $\mathrm{Aut}(\mathcal{S}_{\mathrm{n}})$ | |
|---|---|
| $E$ | $A$ |
| $\phi_1$ | |

$$\xrightarrow{\Delta}$$

| $\mathrm{Aut}(\mathcal{S}_{\mathrm{n}})$ |
|---|
| $\phi_1$ |

Using the element $\Delta$ in $\mathrm{End}_2(\mathcal{S}_{\mathrm{n}})$ defined by:

$$\Delta : \omega \to \begin{cases} \omega & \text{if } \omega \in \mathrm{Aut}(\mathcal{S}_{\mathrm{n}}) \\ \phi_1 & \text{if } \omega = \phi_t \end{cases}$$

Now we can act like $\mathrm{End}(\mathcal{S}_{\mathrm{n}})$ on the image of $\Delta$.

## Tower of $\mathcal{S}_{\mathrm{n}}$ - Embeddings

**Define:** $\Phi_g \in \mathrm{End}_2(\mathcal{S}_{\mathrm{n}})$ as follows:

$$\Phi_g : \omega \to \begin{cases} \psi_g & \text{if } \omega = \psi_t \text{ and } t \in \mathcal{S}_{\mathrm{n}} \setminus \mathcal{A}_{\mathrm{n}} \\ \psi_1 & \text{if } \omega = \psi_t \text{ and } t \in \mathcal{A}_{\mathrm{n}} \\ \phi_1 & \text{otherwise} \end{cases}$$

where $g^2 = 1$

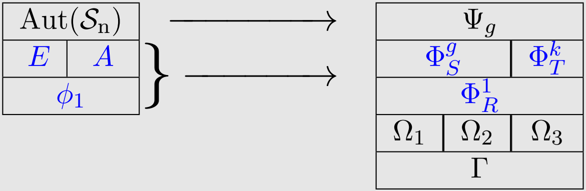
**Recall:** The units of $\mathrm{End}_2(\mathcal{S}_{\mathrm{n}})$: $\Psi_g : \omega \to \psi_g^{-1}\,\omega\,\psi_g$

**The Embedding:**

$$\mathfrak{E} : \mathrm{End}(\mathcal{S}_{\mathrm{n}}) \longrightarrow \mathrm{End}_2(\mathcal{S}_{\mathrm{n}}); \begin{cases} \psi_g \longmapsto \Psi_g \\ \phi_g \longmapsto \Phi_g \end{cases}$$

# Tower of $\mathcal{S}_n$ - Embeddings

**Visually:** $\mathfrak{E} : \mathrm{End}(\mathcal{S}_n) \hookrightarrow \mathrm{End}_2(\mathcal{S}_n)$

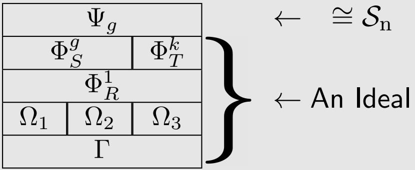



So we have:

$$\mathcal{S}_n \hookrightarrow \mathrm{End}(\mathcal{S}_n) \hookrightarrow \mathrm{End}_2(\mathcal{S}_n)$$

## Tower of $\mathcal{S}_\text{n}$ - Embeddings

**In Fact:** $\mathfrak{F} : \text{End}(\mathcal{S}_\text{n}) \hookrightarrow \text{End}_3(\mathcal{S}_\text{n})$




**Expanding on the idea yields:** $\text{End}_2(\mathcal{S}_\text{n}) \hookrightarrow \text{End}_3(\mathcal{S}_\text{n})$
**Thus:**

$$\mathcal{S}_\text{n} \hookrightarrow \text{End}(\mathcal{S}_\text{n}) \hookrightarrow \text{End}_2(\mathcal{S}_\text{n}) \hookrightarrow \text{End}_3(\mathcal{S}_\text{n}).$$

# Tower of $\mathcal{S}_{\mathrm{n}}$ - Potential Patterns

For the group of units, we have:

| Monoid | $\mathcal{S}_{\mathrm{n}}$ | $\mathrm{End}(\mathcal{S}_{\mathrm{n}})$ | $\mathrm{End}_2(\mathcal{S}_{\mathrm{n}})$ | $\mathrm{End}_3(\mathcal{S}_{\mathrm{n}})$ | $\mathrm{End}_4(\mathcal{S}_{\mathrm{n}})$ |
|---|---|---|---|---|---|
| Group of Units | $\mathcal{S}_{\mathrm{n}}$ | $\cong \mathcal{S}_{\mathrm{n}}$ | $\cong \mathcal{S}_{\mathrm{n}}$ | $\cong \mathcal{S}_{\mathrm{n}}$ | $\cong \mathcal{S}_{\mathrm{n}} \times K$ |

**Question:** Do the units stabilise?

$$\mathrm{Aut}(\mathrm{End}_i(\mathcal{S}_{\mathrm{n}})) \cong \mathcal{S}_{\mathrm{n}} \quad \forall\, i \in \mathbb{N}?$$

For embeddings, we have:

$$\mathcal{S}_{\mathrm{n}} \hookrightarrow \mathrm{End}(\mathcal{S}_{\mathrm{n}}) \hookrightarrow \mathrm{End}_2(\mathcal{S}_{\mathrm{n}}) \hookrightarrow \mathrm{End}_3(\mathcal{S}_{\mathrm{n}})$$

**Question:** Does each level embed into the next?

$$\mathrm{End}_i(\mathcal{S}_{\mathrm{n}}) \hookrightarrow \mathrm{End}_{i+1}(\mathcal{S}_{\mathrm{n}}) \quad \forall\, i \in \mathbb{N}?$$

## Alternative Tower Problems

Let $M$ be a monoid and construct a sequence (The Tower) of endomorphisms

$$M, \quad \mathrm{End}_1(M), \quad \mathrm{End}_2(M), \quad \mathrm{End}_3(M), \quad \cdots$$

**Question:** Do the units stabilise?

$$\mathrm{Aut}(\mathrm{End}_i(M)) \cong U(M) \quad \forall\, i \in \mathbb{N}?$$

**Question:** Does each level embed into the next?

$$\mathrm{End}_i(M) \hookrightarrow \mathrm{End}_{i+1}(M) \quad \forall\, i \in \mathbb{N}?$$